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Abelian varieties and p-divisible groups of Minkowski's conjecture concerning critical lattices of the region, its characteristic $p > 0$ analogues and expansions

Nikolaj Glazunov

We investigate algebraic aspects of the Minkowski's conjecture [**4, 2**] concerning admissible and critical lattices of the region $|x|^p + |y|^p < 1,\ p > 1$, its extensions, and reductions to finite fields. These aspects include lattices (modules), abelian varieties (elliptic curves), p-divisible and formal groups and corresponding moduli spaces. For notations and definitions from geometry of numbers see [**4, 1, 2, 3**]. Recall some of them. Let $\mathcal{R} \subset \mathbb{R}^n$ be a set and Λ be a lattice with the basis $\{a_1, \ldots, a_n\}$ in $\mathbb{R}^n$. A lattice Λ is *admissible* for the body $\mathcal{R}$ ($\mathcal{R}-$*admissible*) if $\mathcal{D} \bigcap \Lambda = \emptyset$ or 0. Let $d(\Lambda)$ be the determinant of Λ. The infimum $\Delta(\mathcal{R})$ of determinants of all lattices admissible for $\mathcal{R}$ is called *the critical determinant* of $\mathcal{R}$; if there is no $\mathcal{R}-$admissible lattices then puts $\Delta(\mathcal{R}) = \infty$. A lattice Λ is *critical* if $d(\Lambda) = \Delta(\mathcal{R})$. Let $D_p = \{(x, y),\ p > 1\} \subset \mathbb{R}^2$ be the 2-dimension region:

$$|x|^p + |y|^p < 1. \tag{1}$$

At first present the Minkowski-Cohn moduli space framework of our considerations of admissible and critical lattices. Let

$$\Delta(p, \sigma) = (\tau + \sigma)(1 + \tau^p)^{-\frac{1}{p}}(1 + \sigma^p)^{-\frac{1}{p}}, \tag{2}$$

be the function defined in the domain

$$\mathcal{M}:\ \infty > p > 1,\ 1 \le \sigma \le \sigma_p = (2^p - 1)^{\frac{1}{p}},$$

of the $\{p, \sigma\}$ plane, where σ is some real parameter; here $\tau = \tau(p, \sigma)$ is the function uniquely determined by the conditions

$$A^p + B^p = 1,\ 0 \le \tau \le \tau_p,$$

where

$$A = A(p, \sigma) = (1 + \tau^p)^{-\frac{1}{p}} - (1 + \sigma^p)^{-\frac{1}{p}},$$

$$B = B(p, \sigma) = \sigma(1 + \sigma^p)^{-\frac{1}{p}} + \tau(1 + \tau^p)^{-\frac{1}{p}},$$

τ_p is defined by the equation

$$2(1 - \tau_p)^p = 1 + \tau_p^p,\ 0 \le \tau_p \le 1.$$

Remark 1. The function $\Delta(p, \sigma)$ in region $\mathcal{M}$ determines the moduli space of admissiblel lattices of the rigion D_p each of which contains three pairs of points on the boundary of D_p.

Theorem 1. *Admissible lattices of the rigion D_p each of which contains three pairs of points on the boundary of D_p are given by lattice generators $a_1 = (x_1, y_1), a_2 = (x_2, y_2)$, where*

$$x_1 = \frac{1}{(1+\tau^p)^{1/p}},\ y_1 = \frac{\tau}{(1+\tau^p)^{1/p}}; \tag{3}$$

$$x_2 = \frac{1}{(1+\sigma^p)^{1/p}},\ y_2 = \frac{\sigma}{(1+\sigma^p)^{1/p}}. \tag{4}$$

Remark 2. Let $r > 1$ be a real and $n > 2$ be an natural. In cases of regions

$$D_r^n : |x_1|^r + |x_2|^r \cdots + |x_n|^r < 1, \tag{5}$$

for $r > 1,\ n > 2$ it is possible to prove similar statements about admissiblel lattices of the regions.

Let $\Delta(D_p)$ be the critical determinant of the region. Let $\Lambda_p^{(0)}$ and $\Lambda_p^{(1)}$ be two D_p-admissible lattices each of which contains three pairs of points on the boundary of D_p and with the property that $(1,0) \in \Lambda_p^{(0)}$, $(-2^{-1/p}, 2^{-1/p}) \in \Lambda_p^{(1)}$, (under these conditions the lattices are uniquely defined). Let $d(\Lambda_p^{(0)}), d(\Lambda_p^{(1)})$ be determinants of the lattices. Let $\Delta_p^{(1)} = \Delta(p,1) = 4^{-\frac{1}{p}}\frac{1+\tau_p}{1-\tau_p}$, $\Delta_p^{(0)} = \Delta(p, \sigma_p) = \frac{1}{2}\sigma_p$.

Remark 3. $d(\Lambda_p^{(0)}) = \Delta(p, \sigma_p),\ d(\Lambda_p^{(1)}) = \Delta(p, 1)$.

Remark 4. For example in the case $p = 2$ the lattice $\Lambda_2^{(0)}$ has the determinant $d(\Lambda_2^{(0)}) = \frac{\sqrt{3}}{2}$ and is defined by generators $a_1 = (1, 0), a_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$.

Theorem 2. **[2]**

$$\Delta(D_p) = \begin{cases} \Delta(p,1),\ 1 < p \le 2,\ p \ge p_0, \\ \Delta(p,\sigma_p),\ 2 \le p \le p_0; \end{cases}$$

here p_0 is a real number that is defined unique by conditions $\Delta(p_0, \sigma_p) = \Delta(p_0, 1),\ 2,57 \le p_0 \le 2,58$.

After complexification of admissible and critical lattices we construct respective elliptic curves and their formal and $p-$divisible groups.

Remark 5. For lattices with pair number of generators $(n = 2m)$ in the case of Remark 2 we obtain after complexification and polarization of lattices corresponding abelian varieties and $p-$divisible groups.

Let now p be a prime number and m be a natural number. We plan to present results about Jacobian varieties and $p-$divisible groups of curves $x^m + y^m = 1$ over prime finite fields $\mathbb{F}_p$ and over their algebraic extensions. We also plan to consider some cases of algebraic hypersurfaces $x_1^r + x_2^r \cdots + x_n^r = 1$ over $\mathbb{F}_p$ when r, n are natural numbers and $n > 2$ with the spesial emphasizes on the case $r = 2$.

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Geometry of numerical series and two-symbol systems of encoding of real numbers

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Let

$$\sum a_n = a_1 + a_2 + \ldots + a_n + r_n = s_n + r_n \tag{1}$$

be an absolutely convergent series. The number x defined by subset $M \subset N$ is called the incomplete sum or subsum of series (1) and is denoted by $x(M)$. It is evident that

$$x(M) = \sum_{n=1}^{\infty} \varepsilon_n a_n \equiv \Delta_{\varepsilon_1 \varepsilon_2 \ldots \varepsilon_n}, \quad \text{where} \quad \varepsilon_n = \begin{cases} 1 & \text{if } n \in M, \\ 0 & \text{if } n \notin M. \end{cases}$$

If M runs over the set of all subsets of the set of positive integers N, we have the numerical set E, i.e.,

$$E[a_n] = \{x : \ x(M), \ M \subset N\}.$$

This set is called the set of incomplete sums of series (1).

Now topological and metric types of the sets of incomplete sums of convergent series are well known. However, criteria for these sets to be nowhere dense sets, Cantorvals or sets of zero measure are not found yet.

Some series provide a basis for two-symbol systems of encoding (representation) for real numbers (with a zero or non-zero redundancy), but, for some other series, two-symbol alphabet is insufficient for creation of a numeral system and should be expanded. This is a topic of this talk. The main object is a polybase two-symbol numeral system with a zero redundancy such that one its base is positive and other one is negative ($0 < q_0 < 1$, $q_1 \equiv q_0 - 1$). We discuss some facts related to geometry of representation, normal properties of numbers in their representations as well as some algebraic aspects of this topic. In particular, we consider groups of transformations preserving tails of "representations", frequencies of digits, etc. Left and right shift operators, inversor of digits, various metrizations of the space of representations form a basis for ergodic theory corresponding to this numeral system. Problems of probabilistic number theory are also considered.

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