(z,k)-equivalence of matrices over Euclidean quadratic rings and solutions of matrix equation $AX+YB=C$  

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Let $K = \mathbb{Z}\left[\sqrt{k}\right]$ be a Euclidean quadratic ring, $e(a)$ be the Euclidean norm $a \in K$ [1].

**Definition 1.** Matrices $A, B \in M(n, K)$ are called (z,k)-equivalent if there exist invertible matrices $S \in GL(n, \mathbb{Z})$ over the ring of integers $\mathbb{Z}$ and $Q \in GL(n, K)$ over quadratic ring $K$ such that $A = SQB$.

We established the standard form of matrices over a Euclidean quadratic ring with respect to the (z,k)-equivalence and used it to the description of the structure of solutions of the matrix equation $AX + YB = C$.

**Theorem 1.** Let $D^A = \text{diag} (\mu_1^A, ..., \mu_n^A)$ be the Smith normal form of a matrix $A$. Then the matrix $A$ is (z,k)-equivalent to the triangular form $T^A$ with invariant factors $\mu_i^A$, $i = 1, ..., n$ on the main diagonal that is

$$SAQ = T^A = TDA,$$

where $T = \left\|t_{ij}\right\|_1^n$ is the lower unitriangular matrix namely $t_{ij} = 0$ if $i < j$, $t_{ij} = 1$ if $i = j$ and $t_{ij} = 0$ if $\mu_i^A = 1$; $e(t_{ij}) < e(\mu_i^A)$ for $t_{ij} \neq 0$, $i, j = 1, ..., n, i > j$.

If $K$ is a Euclidean imaginary quadratic ring, then the matrix $A$ has a finite number of triangular form $T^A$ in the form (1) with respect to (z,k)-equivalence.

Consider the matrix equation

$$AX + YB = C,$$

where $A, B, C \in M(n, K)$ are given matrices and $X, Y \in M(n, K)$ are unknown matrices. Let pair of matrices $(A, B)$ be the (z,k)-equivalent to the pair $(T^A, T^B)$ of matrices $T^A$ and $T^B$ in the form (1) that is $SAQ_A = T^A$, $SBQ_B = T^B$, $S \in GL(n, Z)$, $Q_A, Q_B \in GL(n, K)$ [2]. Then from the equation (2) we get the equation

$$T^AH + WT^B = \tilde{C},$$

where $H = Q_A^{-1}XQ_B$, $W = SYS^{-1}$, $\tilde{C} = SCQ_B$. The matrix equations (2) and (3) are equivalent. Thus the description of the solutions of equation (2) are reduced to the description of the solutions of equation (3).

**Theorem 2.** If the equation (3) has a solution then it has such solutions $H = \left\|h_{ij}\right\|_1^n$, $W = \left\|w_{ij}\right\|_1^n$ that $h_{ij} = 0$ if $\mu_i = 1$, and $e(h_{ij}) < e(\mu_i^B)$ if $h_{ij} \neq 0$, $i, j = 1, ..., n$.

If $K$ is a Euclidean imaginary quadratic ring, then the equation (3) has a finite number of such solutions.
Separability of the lattice of $\tau$-closed totally $\omega$-composition formations of finite groups

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All groups considered are finite. All notations and terminologies are standard [1]-[5].

Let $\omega$ be a non-empty set of primes. Every function of the form $f : \omega \cup \{\omega'\} \to \{\text{formations}\}$ is called an $\omega$-composition satellite. For any $\omega$-composition satellite $CF_\omega(f) = \{G| G/R_\omega(G) \in f(\omega') \text{ and } G/Cp(G) \in f(p) \text{ for all } p \in \pi(\text{Comm}(G)) \cap \omega\}$. If the formation $\mathfrak{F}$ is such that $\mathfrak{F} = CF_\omega(f)$ for some $\omega$-composition satellite $f$, then it is $\omega$-composition formation, and $f$ is $\omega$-composition satellite of this formation.

Every formation of groups is called $\theta$-multiply $\omega$-composition. For $n \geq 1$, a formation $\mathfrak{F}$ is called $n$-multiply $\omega$-composition, if it has an $\omega$-composition satellite $f$ such that every value $f(p)$ of $f$ is an $(n-1)$-multiply $\omega$-composition formation. A formation $\mathfrak{F}$ is called totally $\omega$-composition if it is $n$-multiply $\omega$-composition for all natural $n$.

Let for any group $G$, $\tau(G)$ be a set of subgroups of $G$ such that $G \in \tau(G)$. Then we say following [5] that $\tau$ is a subgroup functor if for every epimorphism $\varphi : A \to B$ and any groups $H \in \tau(A)$ and $T \in \tau(B)$ we have $H^\varphi \in \tau(B)$ and $T^\varphi^{-1} \in \tau(A)$. A class $\mathfrak{F}$ of groups is called $\tau$-closed if $\tau(G) \subseteq \mathfrak{F}$ for all $G \in \mathfrak{F}$.

Let $\mathfrak{X}$ be some set of groups. Then $c_{\omega\infty}^\tau$ form $\mathfrak{X}$ denotes the totally $\omega$-composition formation generated by $\mathfrak{X}$, i.e. $c_{\omega\infty}^\tau$ form $\mathfrak{X}$ is the intersection of all $\tau$-closed totally $\omega$-composition formations containing $\mathfrak{X}$. For any two $\tau$-closed totally $\omega$-composition formations $\mathfrak{M}$ and $\mathfrak{F}$, we write $\mathfrak{M} \lor_{\omega\infty}^\tau \mathfrak{F} = c_{\omega\infty}^\tau$ form($\mathfrak{M} \cup \mathfrak{F}$).

With respect to the operations $\lor_{\omega\infty}^\tau$ and $\cap$ set $c_{\omega\infty}^\tau$ of all $\tau$-closed totally $\omega$-composition formations forms a complete lattice. Formations in $c_{\omega\infty}^\tau$ are called $c_{\omega\infty}^\tau$-formations.

Let $\mathfrak{X}$ be a non-empty class of finite groups. A complete lattice $\theta$ of formations is called $\mathfrak{X}$-separable, if for every term $\nu(x_1, \ldots, x_n)$ of signature $\{\cap, \lor_\theta\}$, $\theta$-formations $\mathfrak{F}_1, \ldots, \mathfrak{F}_n$ and every group $A \in \mathfrak{X} \cap \nu(\mathfrak{F}_1, \ldots, \mathfrak{F}_n)$ are exists $\mathfrak{X}$-groups $A_1 \in \mathfrak{F}_1, \ldots, A_n \in \mathfrak{F}_n$ such that $A \in \nu(\theta\text{form}A_1, \ldots, \theta\text{form}A_n)$. In particular, if $\mathfrak{X} = \mathfrak{G}$ is the class of all finite groups then the lattice $\theta$ of formations is called $\mathfrak{G}$-separable or separable.

**Theorem 1.** The lattice $c_{\omega\infty}^\tau$ all $\tau$-closed totally $\omega$-composition formations is $\mathfrak{G}$-separable.

Let $\tau$ be the trivial subgroup functor or let $\omega$ be the set of all primes. Then we obtain