

# Integer solutions of implicit linear difference equations of the second order

VSEVOLOD MARTSENYUK, SERGIY GEFTER, ALEKSEY PIVEN'

We consider the next implicit linear homogeneous difference equation of the second order:  $cx_{n+2} = bx_{n+1} + ax_n$ ,  $n = 0, 1, 2, \dots$ , where  $a, b, c$  are known integers ( $n = 0, 1, 2, \dots$ ),  $a \neq 0$ ,  $c \geq 2$  and  $b$  or  $a$  is not divisible by  $c$ .

The following main results are obtained.

**THEOREM 1.** *The equation  $cx_{n+2} = bx_{n+1} + ax_n$ ,  $n = 0, 1, 2, \dots$  has only trivial integer solution if and only if the characteristic equation  $c\lambda^2 - b\lambda - a = 0$  has no integer roots.*

**COROLLARY 1.** *Let  $f \in \mathbb{Z}$  and the characteristic equation has no integer roots. Then  $c - b - a \neq 0$  and the equation  $cx_{n+2} = bx_{n+1} + ax_n - f$ ,  $n = 0, 1, 2, \dots$ , has an integer solution if  $c - a - b$  is a divisor of  $f$ . This solution is unique, it is constant and has the form  $x_n = \frac{f}{a+b-c}$ ,  $n = 0, 1, 2, \dots$*

**COROLLARY 2.** *Let a sequence of integer numbers  $\{x_n\}$  is a solution of the equation  $cx_{n+2} = bx_{n+1} + ax_n$ . Then there exist integer numbers  $\alpha$  and  $m$ , such that  $x_n = \alpha m^n$ ,  $n = 0, 1, 2, \dots$*

**THEOREM 2.** *Let  $c$  and  $b$  has a common prime divisor  $p$ , but  $p$  is not a divisor of  $a$ . Let  $\lambda_1, \lambda_2$  are the different roots of the characteristic equation. If  $f_n \in \mathbb{Z}$  and the equation  $cx_{n+2} = bx_{n+1} + ax_n - f_n$ ,  $n = 0, 1, 2, \dots$ , has an integer solution then this solution is unique and has the form*

$$x_n = \sum_{k=0}^{\infty} \left( \frac{\lambda_1^{k+1} - \lambda_2^{k+1}}{\lambda_1 - \lambda_2} \right) \frac{(-1)^k c^k}{a^{k+1}} f_{n+k}, \quad n = 0, 1, 2, \dots$$

*All the terms of this series belong to the ring of integer  $p$ -adic numbers  $\mathbb{Z}_p$  and the sum on the right-hand side is a well defined element of  $\mathbb{Z}_p$ .*

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## CONTACT INFORMATION

### Vsevolod Martsenyuk

V. N. Karazin Kharkiv National University, Kharkiv, Ukraine

*Email address:* martsenyukvv@gmail.com

*URL:* <https://martsenyukvv.wixsite.com/martsenyukvv>

### Sergiy Gefter

V. N. Karazin Kharkiv National University, Kharkiv, Ukraine

*Email address:* gefter@karazin.ua

*URL:* <http://puremath.univer.kharkov.ua/~gefter>

**Aleksey Piven'**

V. N. Karazin Kharkiv National University, Kharkiv, Ukraine

Email address: [aleksei.piven@karazin.ua](mailto:aleksei.piven@karazin.ua)

URL: <http://appmath.univer.kharkov.ua/Piven.htm>

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## Homological algebra in degree zero

ALEX MARTSINKOVSKY

The term "homological algebra in degree zero" refers, in the narrow sense of the word, to calculation of the zeroth derived functor of an additive functor between abelian categories. Most people do not realize that this is an interesting problem because the zeroth right derived functor of a Hom functor is the same Hom functor, and the same can be said about the zeroth left derived functor of the tensor product. The situation changes dramatically if those functors are derived on the opposite sides. In fact, the emerging phenomena seem to be rather diverse and widespread. Those include: a new approach to classical torsion and to Bass torsion, a definition of cotorsion (this is a new concept), dualities between torsion and cotorsion, theorems of Watts and Eilenberg, new results in ring and module theory, a number of formulas that extend several of Auslander's formulas from finitely presented modules to arbitrary modules, a new generalization of Tate homology, a connection between module theory and stable homotopy theory, etc. In this talk, I will try and explain some of those results. Most of this talk is based on joint work with Jeremy Russell.

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