Let \((F_1, F_2, F_3)\) be the lexicographical sequence of functional variables of a ternary generalized functional equation \(T_1 = T_2\) of length three. A sequence \((f_1, f_2, f_3)\) of invertible ternary functions defined on a carrier is called a solution of \(T_1 = T_2\) if substituting \(f_1\) for \(F_1\), \(f_2\) for \(F_2\) and \(f_3\) for \(F_3\), we obtain a true proposition \(t_1 = t_2\), i.e., \(t_1 = t_2\) is an identity [2].

The classification theorem of generalized ternary quadratic quasigroup functional equations of length three is given in [3]. There are four non-equivalent functional equations. The general solution of one of them is given in the following theorem. Solutions of the other three equations are formulated in [4].

**Theorem 1.** A triplet \((f_1, f_2, f_3)\) of ternary invertible operations defined on a set \(Q\) is a solution of the functional equation

\[
F_1(F_2(x, y, z), x, u) = F_3(y, z, u)
\]

if and only if there exist binary invertible operations \(\circ, \ast, \circ\) on \(Q\) such that

\[
\begin{align*}
f_1(y, x, u) &= (x \circ y) \ast u, \\
f_2(x, y, z) &= x \circ^r (y \circ z), \\
f_3(y, z, u) &= (y \circ z) \ast u.
\end{align*}
\]

**References**


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**Some conditions for a quasigroup to be a group isotope**

**Fedir Sokhatsky**

A binary quasigroup is a pair \((Q; \circ)\), where \(Q\) is a set called a carrier and \(\circ\) is an invertible binary operation defined on \(Q\), i.e., there exist operations \(\ell\) and \(\circ\) such that for any \(x, y \in Q\)

\[
(x \circ^\ell y) \circ y = x, \quad (x \circ y) \circ y = x, \quad x \circ (x \circ^\ell y) = y, \quad x \circ (x \circ y) = y.
\]

We say that an identity has a group isotope property, if every quasigroup satisfying this identity is isotopic to a group.

**Definition 1.** We say that variables \(x_1, \ldots, x_n\) are isolated in an identity \(\omega = \nu\) by sub-terms \(t_1, \ldots, t_k\), if all appearances in the identity of the variables belong to two of these terms and every variable has one appearance in at least one of the terms.

Let \(x, y, z\) be arbitrary fixed variables. We will write \(t(x, y)\) if the term \(t\) contains the variables \(x\) and \(y\) and does not contain \(z\).

**Theorem 1.** A quasigroup identity has a group isotopic property if three of its variables \(x, y, z\) are isolated by some sub-terms \(t_1(x, y), t_2(x, z), t_3(y, z)\).
For example, each quasigroup satisfying the identity
\[
(u^{n_1}((x \cdot (xu)^{n_2}y) \cdot x^{n_3}) \cdot u^{n_4}) \cdot (v \cdot (z^{n_5}x \cdot zu)v)u) \cdot (y \cdot (zn^{n_6}y)^n) = v
\]
is isotopic to a group. A bracketing in \(u^{n_1}, (xu)^{n_2}\),\ldots does not matter.

References

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Canonical decompositions of solutions of functional equation of generalized mediality

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Let \(Q\) be a set, a mapping \(f : Q^2 \to Q\) is called an invertible binary operation (=function), if it is invertible element in both semigroups \((O_2; \oplus)\) and \((O_2; \circlearrowright)\), where \(O_2\) is the set of all binary operations defined on \(Q\) and
\[
(f \oplus g)(x, y) := f(g(x, y), y), \quad (f \circlearrowright g)(x, y) := f(x, g(x, y)).
\]
The set of all binary invertible functions is denoted by \(\Delta_2\). A functional equation
\[
F_1(F_2(x, y), F_3(u, v)) = F_4(F_5(x, u), F_6(y, v)), \tag{1}
\]
where \(F_1, \ldots, F_6\) are functional variables and \(x, y, u, v\) are individual variables, is called a functional equation of generalized mediality. The equation was solved in [1]. Namely, the following theorem was proved

THEOREM 1. A sequence \((f_1, \ldots, f_6)\) of invertible functions defined on a set \(Q\) is a solution of (1) if and only if there exists a comutative group \((Q; +, 0)\) and bijections \(\alpha_1, \ldots, \alpha_6\) of \(Q\) such that
\[
f_1(x, z) = \alpha_5x + \alpha_6z, \quad f_2(x, y) = \alpha_5^{-1}(\alpha_1x + \alpha_2y), \quad f_3(u, v) = \alpha_6^{-1}(\alpha_3u + \alpha_4v),
\]
\[
f_4(z, y) = \alpha_7z + \alpha_8y, \quad f_5(x, u) = \alpha_7^{-1}(\alpha_1x + \alpha_3u), \quad f_6(y, v) = \alpha_8^{-1}(\alpha_2y + \alpha_4v).
\]
The sequence \((+, \alpha_1, \ldots, \alpha_8)\) will be called a decomposition of the solution \((f_1, \ldots, f_6)\).

Theorem 1 proves that every solution has a decomposition and moreover every sequence uniquely defines a solution of (1). But the same solution may have different decomposition. For example, let \(\theta\) be an arbitrary automorphism of the group \((Q; +)\), it is easy to see that the sequence \((+, \theta\alpha_1, \ldots, \theta\alpha_8)\) defines the same solution of (1).

A decomposition \((+, \alpha_1, \ldots, \alpha_8)\) of a solution of (1) will be called \(0\)-canonical if 0 is a neutral element of the group \((Q; +)\) and \(\alpha_10 = \alpha_30 = \alpha_70 = 0\).

THEOREM 2. Every element \(0 \in Q\) uniquely defines a canonical decomposition of an arbitrary solution of the functional equation of generalized mediality.