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Key words and phrases. Finite group, isoordic groups, σ-nilpotent group, σ-subnormal subgroup.

Necessary and sufficient condition for the existence of one-point time on an oriented set

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Definition 1. The ordered pair $\mathcal{M} = \left( \mathcal{B}\mathcal{s}(\mathcal{M}), \preceq \right)$ is called an oriented set if and only if $\mathcal{B}\mathcal{s}(\mathcal{M})$ is some non-empty set ($\mathcal{B}\mathcal{s}(\mathcal{M}) \neq \emptyset$) and $\preceq$ is arbitrary reflexive binary relation on $\mathcal{B}\mathcal{s}(\mathcal{M})$. In this case the set $\mathcal{B}\mathcal{s}(\mathcal{M})$ is named the basic set or the set of all elementary states of the oriented set $\mathcal{M}$ and the relation $\preceq$ is named by the directing relation of changes (transformations) of $\mathcal{M}$.

In the case where the oriented set $\mathcal{M}$ is known in advance, the char $\mathcal{M}$ in the notation $\preceq$ will be released, and we will use the notation $\preceq$ instead. From an intuitive point of view, oriented sets may be interpreted as the most primitive models of sets of evolving objects.

Definition 2. Let $\mathcal{M}$ be an oriented set and $\mathcal{T} = (\mathcal{T}, \preceq)$ be a linearly ordered set. A mapping $\psi : \mathcal{T} \mapsto 2^{\mathcal{B}\mathcal{s}(\mathcal{M})}$ is referred to as time on $\mathcal{M}$ if the following conditions are satisfied:

1. For any elementary state $x \in \mathcal{B}\mathcal{s}(\mathcal{M})$ there exists an element $t \in \mathcal{T}$ such that $x \in \psi(t)$.
2. If $x_1, x_2 \in \mathcal{B}\mathcal{s}(\mathcal{M})$, $x_2 \preceq x_1$ and $x_1 \neq x_2$, then there exist elements $t_1, t_2 \in \mathcal{T}$ such that $x_1 \in \psi(t_1)$, $x_2 \in \psi(t_2)$ and $t_1 < t_2$ (this means that there is a temporal separateness of successive unequal elementary states).

In this case the elements $t \in \mathcal{T}$ we call the moments of time.

It turns out that any oriented set $\mathcal{M}$ can be chronologized (that is we can define some time on it). To make sure this we may consider any linearly ordered set $\mathcal{T} = (\mathcal{T}, \preceq)$, which contains at least two elements ($\text{card}(\mathcal{T}) \geq 2$) and put, $\psi(t) := \mathcal{B}\mathcal{s}(\mathcal{M})$, $t \in \mathcal{T}$.

Definition 3. Let $\mathcal{M}$ be an oriented set.

a. The time $\psi : \mathcal{T} \mapsto 2^{\mathcal{B}\mathcal{s}(\mathcal{M})}$ is called by quasi one-point if for any $t \in \mathcal{T}$ the set $\psi(t)$ is a singleton.

b. The time $\psi$ is called one-point if the following conditions are satisfied:

(a) the time $\psi$ is quasi one-point;
(b) for every $x_1, x_2 \in \mathcal{B}\mathcal{s}(\mathcal{M})$ the conditions $x_1 \in \psi(t_1)$, $x_2 \in \psi(t_2)$ and $t_1 \leq t_2$, assure the correlation $x_2 \preceq x_1$.

Example 1. Let us consider an arbitrary mapping $f : \mathbb{R} \mapsto \mathbb{R}^d$ ($d \in \mathbb{N}$). This mapping can be interpreted as equation of motion of a single material point in the space $\mathbb{R}^d$. The mapping $f$ generates the oriented set $\mathcal{M}_f = \left( \mathcal{B}\mathcal{s}(\mathcal{M}_f), \preceq \right)$, where $\mathcal{B}\mathcal{s}(\mathcal{M}_f) = \mathcal{R}(f) = \{ f(t) \mid t \in \mathbb{R} \} \subseteq \mathbb{R}^d$ and for $x, y \in \mathcal{B}\mathcal{s}(\mathcal{M})$ the correlation $y \preceq x$ holds if and only if there exist $t_1, t_2 \in \mathbb{R}$ such,
that \( x = f(t_1), \ y = f(t_2) \) and \( t_1 \leq t_2 \). It is easy to verify, that the following mapping is a one-point time on \( \mathcal{M}_f \):

\[
\psi(t) = \{ f(t) \} \subseteq \mathcal{B}s(\mathcal{M}), \quad t \in \mathbb{R}.
\]

Example 1 makes clear the notion of one-point time. It is evident, that any one-point time is quasi one-point. There exist the counterexamples, which show that the inverse statement, in general, is not true.

**Theorem 1 (see [1, 2]).** Any oriented set \( \mathcal{M} \) can be quasi one-point chronologized (this means that we can define some quasi one-point time on \( \mathcal{M} \)).

On any oriented set \( \mathcal{M} \) we introduce the following additional binary relation:

a. For every \( x, y \in \mathcal{B}s(\mathcal{M}) \) we note \( y \xleftarrow{\mathcal{M}} x \) if and only if \( y \xleftarrow{\mathcal{M}} x \) and \( x \not\xleftarrow{\mathcal{M}} y \).

b. In the cases where it does not lead to misunderstanding we use the notation \( y \xleftarrow{\mathcal{M}} x \) instead of the record \( y \xleftarrow{\mathcal{M}} x \).

**Definition 4.** The oriented set \( \mathcal{M} \) is called **quasi-chain** if and only if the following conditions are satisfied:

- For any \( x_1, x_2 \in \mathcal{B}s(\mathcal{M}) \) it holds at least one from the correlations \( x_2 \xleftarrow{\mathcal{M}} x_1 \) or \( x_1 \xleftarrow{\mathcal{M}} x_2 \).
- For every \( x_0, x_1, x_2, x_3 \in \mathcal{B}s(\mathcal{M}) \) the conditions \( x_3 \xleftarrow{\mathcal{M}} x_2, \ x_2 \xleftarrow{\mathcal{M}} x_1 \) and \( x_1 \xleftarrow{\mathcal{M}} x_0 \) lead to the correlation \( x_3 \xleftarrow{\mathcal{M}} x_0 \) (quasitransitivity).

It is easy to prove that the transitivity of the binary relation \( \xleftarrow{\mathcal{M}} \) on the oriented set \( \mathcal{M} \) implies its quasitransitivity. It can be proven that the inverse statement in general is not valid. That is there exist the oriented set \( \mathcal{M} \) such that the relation \( \xleftarrow{\mathcal{M}} \) is quasitransitive but not transitive.

The main result of the talk is the following theorem, which gives the necessary and sufficient condition of existence for one-point time on the oriented set.

**Theorem 2 (ZF+AC).** The oriented \( \mathcal{M} \) set can be one-point chronologized if and only if it is quasi-chain.

We emphasize that proof of the necessity for Theorem 2 does not require the axiom of choice (AC). This axiom is needed only for the proof of sufficiency of the condition, pointed out in Theorem 2.

**References**


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*Key words and phrases.* Oriented sets, changeable sets, time, ordered sets

This research was partially supported by Budget Program “Support to the development of priority areas of scientific research” KPKVK 6541230.