Closure operators in Morita contexts: mappings and their properties

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Closure operator of a module category $R$-$\text{Mod}$ is a function $C$, which associates to every submodule $N \subseteq M$, where $M \in R$-$\text{Mod}$, a submodule $C_M(N) \subseteq M$, which satisfies the conditions of extension, monotony and continuity ([2, 3]). Denote by $\mathfrak{C}(R)$ the class of all closure operators of $R$-$\text{Mod}$.

Let $(R, R U_S, S V_R, S)$ be an arbitrary Morita context with the morphisms $(\cdot) : U \otimes_R V \to R$ and $[\cdot] : V \otimes_R U \to S ([1])$. We consider the functors $R$-$\text{Mod} \xrightarrow{H^V=\text{Hom}_R(U, \cdot)} S$-$\text{Mod}$ with the associated natural transformations $\varphi : 1_{R-$-$\text{Mod}} \to H^V H^U$ and $\psi : 1_{S-$-$\text{Mod}} \to H^V H^U$.

The purpose of this study is to establish the relation between the classes of closure operators $\mathfrak{C}(R)$ and $\mathfrak{C}(S)$ determined by the functors $H^V$ and $H^U$ for the given Morita context $(R, R U_S, S V_R, S)$. For that two mappings are constructed $\mathfrak{C}(R) \xrightarrow{(-)^*} \mathfrak{C}(S)$ between the classes of closure operators. The transition $C \rightsquigarrow C^*$, where $C \in \mathfrak{C}(R)$, is defined by the rule: $(C)^*_Y (N) \overset{\text{def}}{=} \ker \left[ \psi_Y \cdot H^U(\pi^n_C) \right]$, where $n : N \xrightarrow{\approx} Y$ is an inclusion of $S$-$\text{Mod}$ and $\pi^n_C : H^Y(Y) \to H^Y(Y) \backslash C_{H^Y(Y)}(\text{Im} H^V(n))$ is a natural epimorphism. Similarly, $D \rightsquigarrow D^*$ is defined for $D \in \mathfrak{C}(S)$.

Some important properties of “star” mappings are proved. In particular:

1) the “star” mappings are monotone, i.e. $C_1 \leq C_2 \Rightarrow C_1^* \leq C_2^*$ and $D_1 \leq D_2 \Rightarrow D_1^* \leq D_2^*$;
2) $C \leq C^{**}$ for every $C \in \mathfrak{C}(R)$, $D \leq D^{**}$ for every $D \in \mathfrak{C}(S)$;
3) $(\bigwedge_{\alpha \in \mathfrak{A}} C_\alpha)^* = \bigwedge_{\alpha \in \mathfrak{A}} C^*_\alpha$ for every family $\{C_\alpha \mid \alpha \in \mathfrak{A}\} \subseteq \mathfrak{C}(R)$;
4) $(\bigwedge_{\alpha \in \mathfrak{A}} D_\alpha)^* = \bigwedge_{\alpha \in \mathfrak{A}} D^*_\alpha$ for every family $\{D_\alpha \mid \alpha \in \mathfrak{A}\} \subseteq \mathfrak{C}(S)$.

References


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