Cramer’s rules for Sylvester-type quaternion matrix equations

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Consider the two-sided generalized Sylvester matrix equation

\[ AXB + CYD = E \]  

(1)

over the quaternion skew field \( \mathbb{H} \). The Sylvester matrix equation has far reaching applications in different fields (see, e.g., [1]). Its solving is based on generalized inverses which are important tools in solving of matrix equations. Let for \( A \in \mathbb{H}^{m \times n} \), \( A^\dagger \) mean its Moore-Penrose generalized inverse, i.e. the exclusive matrix \( X \in \mathbb{H}^{n \times m} \) satisfying \( AXA = A \), \( XAX = X \), \( (AX)^* = AX \), \( (XA)^* =XA \). Furthermore, let \( L_A = I - A^\dagger A \) and \( R_A = I - AA^\dagger \) be a couple of projectors induced by \( A \). In [2] the solvability conditions to Eq. (1) was obtained and its general solution was expressed in terms of generalized inverses as follows:

\[ X = A^\dagger EB^\dagger - A^\dagger CM^\dagger R_A EB^\dagger - A^\dagger SC^\dagger EL_B N^\dagger DB^\dagger - A^\dagger SVR_N DB^\dagger + L_A U + ZR_B, \]

\[ Y = M^\dagger R_A ED^\dagger + M_S^\dagger SC^\dagger EL_B N^\dagger + M_L(V - S^\dagger SVNN^\dagger) + WR_D, \]

where \( U, V, Z \) and \( W \) are arbitrary matrices of suitable sizes over \( \mathbb{H} \), \( M := R_A C \), \( N := DL_B \), and \( S := CL_M \).

Using determinantal representations of the Moore-Penrose inverse, previously obtained in [3], within the framework of the theory of quaternion row-column determinants (introduced in [4, 5]), we got in [6] explicit determinantal representation formulas (analogs of Cramer’s Rule) for the solution to Eq. (1) and to its special cases when its first term or both terms are one-sided. The Cramer’s Rules for general, Hermitian, or \( \eta \)-Hermitian solutions (\( \eta \in \{i, j, k\} \)) to the Sylvester-type matrix equations involving \(*\)-Hermicity or \(\eta\)-Hermicity (i.e. when in Eq. (1), \( B = A^* \) and \( D = C^* \), or \( B = A^{\eta^*} \) and \( D = C^{\eta^*} \), respectively) are derived in [7].

References
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(z,k)-equivalence of matrices over Euclidean quadratic rings and solutions of matrix equation AX+YB=C

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Let $K = \mathbb{Z}\left[\sqrt{k}\right]$ be a Euclidean quadratic ring, $e(a)$ be the Euclidean norm $a \in K$ [1].

Definition 1. Matrices $A, B \in M(n, K)$ are called (z,k)-equivalent if there exist invertible matrices $S \in GL(n, \mathbb{Z})$ over the ring of integers $\mathbb{Z}$ and $Q \in GL(n, K)$ over quadratic ring $K$ such that $A = SBQ$.

We established the standard form of matrices over a Euclidean quadratic ring with respect to the (z,k)-equivalence and used it to the description of the structure of solutions of the matrix equation $AX + YB = C$.

Theorem 1. Let $D^A = \text{diag}(\mu_1^A, ..., \mu_n^A)$ be the Smith normal form of a matrix $A$. Then the matrix $A$ is (z,k)-equivalent to the triangular form $T^A$ with invariant factors $\mu_i^A$, $i = 1, ..., n$ on the main diagonal that is

$$SAQ = T^A = TD^A, \quad S \in GL(n, \mathbb{Z}), \quad Q \in GL(n, K)$$

where $T = \left\|t_{ij}\right\|_1^n$ is the lower unitriangular matrix namely $t_{ij} = 0$ if $i < j$, $t_{ij} = 1$ if $i = j$ and $t_{ij} = 0$ if $\mu_i^A = 1$; $e(t_{ij}) < e(\mu_i^A)$ for $t_{ij} \neq 0$, $i, j = 1, ..., n$, $i > j$.

If $K$ is a Euclidean imaginary quadratic ring, then the matrix $A$ has a finite number of triangular form $T^A$ in the form (1) with respect to (z,k)-equivalence.

Consider the matrix equation

$$AX + YB = C,$$  \hspace{1cm} (2)

where $A, B, C \in M(n, K)$ are given matrices and $X, Y \in M(n, K)$ are unknown matrices. Let pair of matrices $(A, B)$ be the (z,k)-equivalent to the pair $(T^A, T^B)$ of matrices $T^A$ and $T^B$ in the form (1) that is $SAQ_A = T^A$, $SBQ_B = T^B$, $S \in GL(n, \mathbb{Z})$, $Q_A, Q_B \in GL(n, K)$ [2]. Then from the equation (2) we get the equation

$$T^A H + W T^B = \tilde{C},$$  \hspace{1cm} (3)

where $H = Q_A^{-1} X Q_B$, $W = SY S^{-1}$, $\tilde{C} = SCQ_B$. The matrix equations (2) and (3) are equivalent. Thus the description of the solutions of equation (2) are reduced to the description of the solutions of equation (3).

Theorem 2. If the equation (3) has a solution then it has such solutions $H = \left\|h_{ij}\right\|_1^n$, $W = \left\|w_{ij}\right\|_1^n$ that $h_{ij} = 0$ if $\mu_i = 1$, and $e(h_{ij}) < e(\mu_i^B)$ if $h_{ij} \neq 0$, $i, j = 1, ..., n$.

If $K$ is a Euclidean imaginary quadratic ring, then the equation (3) has a finite number of such solutions.