Solvable Lie algebras of derivations of rank one

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Let \( \mathbb{K} \) be a field of characteristic zero and \( A = \mathbb{K}[x_1, \ldots, x_n] \) the polynomial ring over \( \mathbb{K} \). A \( \mathbb{K} \)-derivation \( D \) of \( A \) is a \( \mathbb{K} \)-linear mapping \( D: A \to A \) that satisfies the rule: \( D(ab) = D(a)b + aD(b) \) for all \( a, b \in A \). The set \( W_n(\mathbb{K}) \) of all \( \mathbb{K} \)-derivations of the polynomial ring \( A \) forms a Lie algebra over \( \mathbb{K} \). This Lie algebra is simultaneously a free module over \( A \) with the standard basis \( \left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n} \right\} \). Therefore, for each subalgebra \( L \) of \( W_n(\mathbb{K}) \) one can define the rank \( \operatorname{rank}_AL \) of \( L \) over the ring \( A \). Note that for any \( f \in A \) and \( D \in W_n(\mathbb{K}) \) a derivation \( fD \) is defined by the rule: \( fD(a) = f \cdot D(a) \) for all \( a \in A \).

Finite dimensional subalgebras \( L \) of \( W_n(\mathbb{K}) \) such that \( \operatorname{rank}_AL = 1 \) were described in [1]. We study solvable subalgebras \( L \subseteq W_n(\mathbb{K}) \) of rank 1 over \( A \) without restrictions on the dimension over the field \( \mathbb{K} \).

Recall that a polynomial \( f \in A \) is said to be a Darboux polynomial for a derivation \( D \in W_n(\mathbb{K}) \) if \( f \neq 0 \) and \( D(f) = \lambda f \) for some polynomial \( \lambda \in A \). The polynomial \( \lambda \) is called the polynomial eigenvalue of \( f \) for the derivation \( D \). Some properties of Darboux polynomials and their applications in the theory of differential equations can be found in [3]. Denote by \( A_D^1 \) the set of all Darboux polynomials for \( D \in W_n(\mathbb{K}) \) with the same polynomial eigenvalue \( \lambda \) and of the zero polynomial. Obviously, the set \( A_D^1 \) is a vector space over \( \mathbb{K} \). If \( V \) is a subspace of \( A_D^1 \) for any derivation \( D \in W_n(\mathbb{K}) \), then we denote by \( VD \) the set of all derivations \( fD, f \in V \).

**Theorem 1.** Let \( L \) be a subalgebra of the Lie algebra \( W_n(\mathbb{K}) \) of rank 1 over \( A \) and \( \dim_{\mathbb{K}} L \geq 2 \). The Lie algebra \( L \) is abelian if and only if there exist a derivation \( D \in W_n(\mathbb{K}) \) and a Darboux polynomial \( f \) for \( D \) with the polynomial eigenvalue \( \lambda \) such that \( L = VD \) for some \( \mathbb{K} \)-subspace \( V \subseteq A_D^1 \).

Using this result one can characterize nonabelian subalgebras of rank 1 over \( A \) of the Lie algebra \( W_n(\mathbb{K}) \). For the Lie algebra \( W_n(\mathbb{K}) \) of all \( \mathbb{K} \)-derivations of the field \( \mathbb{K}(x_1, x_2, \ldots, x_n) \) this problem is simpler and was considered in [2].

**References**

Classification of quasigroups according to their parastrophic symmetry groups

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Let \( Q \) be a set, a mapping \( f : Q^3 \to Q \) is called an invertible ternary operation (=function), if it is invertible element in all semigroups \((\mathcal{O}_3; \oplus), (\mathcal{O}_3; \ominus)\) and \((\mathcal{O}_3; \otimes)\), where \( \mathcal{O}_3 \) is the set of all ternary operations defined on \( Q \) and
\[
(f \oplus f_1)(x_1, x_2, x_3) := f(f_1(x_1, x_2, x_3), x_2, x_3), \quad (f \oplus f_1)(x_1, x_2, x_3) := f(x_1, f_1(x_1, x_2, x_3), x_3),
\]
\[
(f \oplus f_1)(x_1, x_2, x_3) := f(x_1, x_2, f_1(x_1, x_2, x_3)).
\]
The set of all ternary invertible functions is denoted by \( \Delta_3 \). If an operation \( f \) is invertible and \((14)f, (24)f, (34)f\) are its inverses in those semigroups, then the algebra \((Q; f, (14)f, (24)f, (34)f)\) (in brief, \((Q; f)\)) is called a ternary quasigroup [1]. The inverses are also invertible. All inverses to inverses are called \( \sigma \)-parastrophes of the operation \( f \) and can be defined by
\[
\sigma f(x_\sigma, x_\sigma, x_\sigma) = x_\sigma \iff f(x_1, x_2, x_3) = x_4, \quad \sigma \in S_4,
\]
where \( S_4 \) denotes the group of all bijections of the set \( \{0, 1, 2, 3\} \). Therefore in general, every invertible operation has 24 parastrophes. Since parastrophes of a quasigroup satisfy the equalities \( \eta(f) = \sigma f \), then the symmetric group \( S_4 \) defines an action on the set \( \Delta_3 \). In particular, this fact implies that the number of different parastrophes of an invertible operation is a factor of 24. More precisely, it is equal to \( 24/|\text{Ps}(f)| \), where \( \text{Ps}(f) \) denotes a stabilizer group of \( f \) under this action which is called parastrophic symmetry group of the operation \( f \).

Let \( \mathfrak{P}(H) \) denote the class of all quasigroups whose parastrophic symmetry group contains the group \( H \in S_4 \). A ternary quasigroup \((Q; f)\) belongs to \( \mathfrak{P}(H) \) if and only if \( \sigma f = f \) for all \( \sigma \) from a set \( G \) of generators of the group \( H \), therefore, the class of quasigroup \( \mathfrak{P}(H) \) is a variety.

For every subgroup \( H \) of the group \( S_4 \) the variety \( \mathfrak{P}(H) \) is described and its subvariety of ternary group isotopes are found. For example, let
\[
D_8 := \{e, (12), (34), (12)(34), (13)(24), (14)(23), (1324), (1423)\} \leq S_4.
\]

Theorem 1. A ternary quasigroup \((Q; f)\) belong to the variety \( \mathfrak{P}(D_8) \) if and only if
\[
f(x, y, z) = f(y, x, z), \quad f(x, y, f(x, y, z)) = z, \quad f(z, f(x, y, z), x) = y.
\]