Reduction of nonsingular matrices over rings of almost stable range 1

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All rings consider of will be a commutative with nonzero units. Recall that a ring $R$ is a Bezout ring if it every finitely generated ideal is principal. A ring $R$ is called an elementary divisor ring if for any $n \times m$ matrix $A$ over $R$ there exist invertible matrices $P \in GL_n(R)$ and $Q \in GL_m(R)$ such that $PAQ = D$ is a diagonal matrix. $D = (d_{ii})$ and $d_{i+1,i+1}R \subseteq d_{ii}R$ [1].

We denote by $GE_n$ the subgroup of $GL_n(R)$ generated by the elementary matrices.

A ring $R$ is called a ring of stable range 1 if for any elements $a, b \in R$ the equality $aR + bR = R$ implies that there is some $x \in R$ such that $(a + bx)R = R$.

DEFINITION 1. An element $a \neq 0$ of a commutative ring $R$ is called an element almost stable range 1 if the stable range of a factor-ring $R/aR$ is equal to 1. If all nonzero elements of a ring $R$ are elements of almost stable range 1 then we say that $R$ is a ring of almost stable range 1.

THEOREM 1. Let $R$ be commutative Bezout domain of almost stable range 1, then for any nonsingular matrix of size $n$ we can find such unimodular matrices $P \in GE_n(R)$ and $Q \in GL_n(R)$, that

$$PAQ = \begin{pmatrix} 
\varepsilon_1 & 0 & \ldots & 0 \\
0 & \varepsilon_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \varepsilon_n 
\end{pmatrix},$$

where $\varepsilon_i$ is divisor $\varepsilon_{i+1}$, $1 \leq i \leq n - 1$.

References

On the conjugate sets of IP-quasigroups

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A quasigroup \((Q, A)\) is called quasigroup with the property of invertibility (an IP-quasigroup) if there exist two mappings \(I_t\) an \(I_r\) on the set \(Q\) into \(Q\) that \(A(I_tx, A(x, y)) = y\) and \(A(A(y, x), I_rx) = y\) for any \(x, y \in Q\) [1]. The mappings \(I_t\) and \(I_r\) are permutations and \(I_t^2 = I_r^2 = \varepsilon\).

It is known that the system \(\Sigma\) of six (not necessarily distinct) conjugates (or parastrophes): \(A, A^r, A^l, r_lA, r_rA, A^s\), where \(A(x, y) = z \Leftrightarrow A(x, z) = y, A(y, x) = z \Leftrightarrow A(z, y) = x, A(x, y) = A(y, x)\) \((r_rA = (QA)\) corresponds to a quasigroup \((Q, A)\).

It is known [2] that the number of distinct conjugates in \(\Sigma\) can be 1, 2, 3 or 6.

Using suitable Belousov's designation of conjugates of a quasigroup \((Q, A)\) from [1] we have the following system \(\Sigma\) of conjugates:

\[
\Sigma = \{A, A^r, A^l, r_lA, r_rA, A^s\},
\]

where \(l^rA = A, l^lA = A^{-1}, l^sA = A^{-1}, l^nA = (-A)^{-1}, l^rA = sA = A^s\).

Note that

\[
(-1(A^{-1}))^{-1} = r_{lr}A = -1((-A)^{-1}) = r_{rl}A = sA
\]

and \(r_rA = yA = A, \sigma yA = \sigma(A^r)\).

The conjugates of IP-quasigroup have the following form [1, 4]: \(l^rA(x, y) = A(x, I_r y), l^nA(x, y) = A(l_x, y), r_lA(x, y) = I_lA(x, I_y), r_rA(x, y) = I_rA(l_x, y)\).

The following Theorem 1 of [3, 4] describes all possible conjugate sets for quasigroups and points out the only possible variants of equality of conjugates:

**Theorem 1.** The following conjugate sets of a quasigroups \((Q, A)\) are only possible:

\[
\Sigma_1 = \{A\}, \Sigma_2 = \{A, A^r\}, \Sigma_3 = \{A = r_lA = r_rA, A = \sigma A = A^s\}, \Sigma_4 = \{A, A^r, A^l, r_lA, r_rA, A^s\}, \Sigma_5 = \{A = r_lA, A = r_rA, r_rA = sA\};
\]

\[
\Sigma_6 = \{A = r_lA, A = r_rA, A = yA, A = sA\}; \Sigma_7 = \{A = r_lA, A = r_rA, A = r_rA = sA\}.
\]

We study the conjugate sets on the distinct conjugates of IP-quasigroups and IP-loops.

**Theorem 2.** Let a quasigroup \((Q, A)\) be an IP-quasigroup. Then

\[
\Sigma(A) = \Sigma_1(A)\] if and only if \(I_r = I = \varepsilon\);
\[
\Sigma(A) = \Sigma_2(A)\] if and only if \(I_r = I, I \neq \varepsilon, A(x, y) \neq A(y, x)\) and \(IA(x, y) = A(y, x)\);
\[
\Sigma(A) = \Sigma_3(A)\] if and only if \(I_r = \varepsilon \neq I\);
\[
\Sigma(A) = \Sigma_4(A)\] if and only if \(I_r = \varepsilon \neq I\);
\[
\Sigma(A) = \Sigma_5(A)\] if and only if \(I_r = \varepsilon \neq I\); and \(A(x, y) = A(y, x)\);
\[
\Sigma(A) = \Sigma_6(A)\] if and only if \(I_r = I \neq \varepsilon\) and \(A(x, y) = A(y, x)\) and \(IA(x, y) = A(y, x)\).