For the limit $q \to q^{-1}$ we get

$$\{n,m\} \equiv \lim_{q\to 1}[n,m]_{q,q} = \lim_{q\to 1}[n,m]_{q,q^{-1}} = \frac{1}{2}(n+1)(m+1)(n+m+2) = \dim \Gamma_{n,m},$$

$$\{n-1,n-1\} = n^3 = \dim \Gamma_{n-1,n-1},$$

$$\{n-1,0\} = \{0,n-1\} = \frac{n(n+1)}{2} = \dim \Gamma_{n-1,0}$$

For $p \to 1$ the $(q,p)$-numbers $[r]_{q,p}$ turn into the Jackson $q$-numbers $[r)_q \equiv (1-q^n)/(1-q)$. We prove that

$$[n,m]_{q,1} = q^{-(n+m)}\frac{[n+m+2)_q[n+1)_q[m+1)_q}{[2]_q},$$

$$[n,m]_{q,1} = q^{-2n}\frac{[n+1)_q^2[2(n+1))_q}{[2]_q},$$

$$[n-1,0]_{q,1} = \frac{q^{-n}[n)_q[n+1)_q}{[2]_q}.$$

### References

1. J. Humphreys, *Introduction to Lie Algebras and Representation Theory*, 1978.
2. W. Fulton, J. Harris, *Reptesentation theory: a first course,* 1991.

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# Some properties of generelized hypergeometric Appell polynomials

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In [**1**], P.Appell presented the sequence of polynomials $\{A_n(x)\}, n = 0, 1, 2, \dots$ which satisfies the following relation

$$A_n'(x) = nA_{n-1}(x),$$

and possesses the exponential generating function

$$A(t)e^{xt} = \sum_{n=0}^{\infty} A_n(x)\frac{t^n}{n!},$$

where $A(t)$ is a formal power series

$$A(t) = a_0 + a_1 t + a_2\frac{t^2}{2!} + \cdots + a_n\frac{t^n}{n!} + \cdots \quad , a_0 \neq 0.$$

The Appell type polynomials appear at the different areas of mathematics, namely, at special functions, general algebra, combinatorics and number theory. Resently, the Appell type polynomials are of big interest. New approaches based on the determinant method and Pascal matrix method are applied (see, e.g., [**2**]–[**3**]).

Monomials $x^n$, Bernoulli polynomials, Euler polynomials and Hermite polynomials are the examples of the Appell type polynomials ([**4**]).

Definition. Let us

$$\Delta(k,-n) = -\frac{n}{k}, -\frac{n-1}{k}, \cdots, -\frac{n-k+1}{k}, \quad k,n \in \mathbb{Z}.$$

Then polynomials $A_n^{(k)}(m,x)$, $n = 0,1,2,\dots$ where

$$A_n^{(k)}(x) = x^n {}_{k+p}F_q \left[ \begin{matrix} \Delta(k,-n), & \alpha_1,\alpha_2,\dots,\alpha_p \\ & \beta_1,\beta_2,\dots,\beta_q \end{matrix} \middle| \frac{m}{x^k} \right],$$

and $m,k \in \mathbb{N}_0$, $\alpha_1,\alpha_2,\dots,\alpha_p$, $\beta_1,\beta_2,\dots,\beta_p$ are the arbitrary number sets, we call *generalized hypergeometric Appell polynomials.*

In the case when $p = 0, q = 0$, $k := m$, $h := \frac{(-1)^k}{k^k}$ the generelized hypergeometric Appell polynomials $A_n^{(k)}(p,q;x)$ became the Gould-Hoppers polynomials [**5**], and with $p = 0, q = 0$ and $k = 2$ they are the well-known Hermite polynomials.

Theorem 1. *Generalized hypergeometric Appell polynomials* $A_n^{(k)}(m,x)$ *are the Appell type polynomials.*

Proof. To prove it one should replace $t \mapsto xt$, $x \mapsto \frac{m}{x^k}$, in problem 26, p.173 [**6**], then the function $A(t)$ takes a form

$$A(t) = {}_pF_q \left[ \begin{matrix} a_1,a_2,\dots,a_p \\ b_1,b_2,\dots,b_q \end{matrix} \middle| (-1)^k m \frac{t^k}{k^k} \right].$$

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Using the derivative properties of the composition of functions and the hypergeometric function we obtain

Theorem 2. *The following identity holds*

$$nx^{k-1} {}_{p+k}F_q \left[ \begin{matrix} a_1,a_2,\dots,a_p,\Delta(k,-n) \\ b_1,b_2,\dots,b_q \end{matrix} \middle| \frac{m}{x^k} \right] =$$

$$= km \frac{a_1 a_2 \dots a_p}{b_1 b_2 \cdots b_q} \Delta_1(k,-n) {}_{p+k}F_q \left[ \begin{matrix} a_1+1,a_2+1,\dots,a_p+1,\Delta(k,-n)+1 \\ b_1+1,b_2+1,\dots,b_q+1 \end{matrix} \middle| \frac{m}{x^k} \right] +$$

$$+ nx^k {}_{p+k}F_q \left[ \begin{matrix} a_1,a_2,\dots,a_p,\Delta(k,-n+1) \\ b_1,b_2,\dots,b_q \end{matrix} \middle| \frac{m}{x^k} \right],$$

*where*

$$\Delta_1(k,-n) = \left(-\frac{n}{k}\right) \cdot \left(-\frac{n-1}{k}\right) \cdots \left(-\frac{n-k+1}{k}\right).$$

Further on, the generalized hypergeometric Appell polynomials possess the convolution type property.

Theorem 3.

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} A_i^{(k)}(m,0) A_{n-i}^{(k)}(m,0) = 2^{\frac{n}{k}} A_n^{(k)}(m,0).$$

Proof. The proof is based on the method proposed in [**7**]. ☐

## References

1. P. Appell, *Sur une classe de polynomes* [*On one class of polynomials*], Annales scientifiques de l'E.N.S. 2e serie, tome 9, 1880, 119-144.
2. Y. Yang, C. Micek, *Generalized Pascal functional matrix and its applications*, Linear Algebra Appl. 423 (2007), 230–245.
3. F. A. Costabile, E. Longo. *A determinantal approach to Appell polynomials*, Journal of Computational and Applied Mathematics Volume 234, Issue 5, 1 July 2010, 1528-1542.
4. M. Hazewinkel, *Appell polynomials*, Encyclopedia of Mathematics, Springer Science+Business Media B.V. / Kluwer Academic Publishers, (2001) [1994]. ISBN 978-1-55608-010-4.
5. H. W. Gould, A. T. Hopper *Operational formulas connected with two generalizations of Hermite Polynomials*, Duke Math. J., 29 (1962), 51–62.
6. H. M. Srivastava, H. L. Manocha, *A treatise on generating functions*, Horwood, 1984, 569 p.
7. Y. Ben Cheikh, H. Chaggara, *Connection problems via lowering operators*, Journal of Computational and Applied Mathematics Volume 178, Issues 1–2, 1 June 2005, 45-61.

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# Tensor products of indecomposable integral matrix representations of the symmetric group of third degree

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Let $S_3$ be the symmetric group of third degree with generators $a$, $b$ and relations: $a^2 = b^3 = e$, $ba = ab^2$, where $e$ is the identity of $S_3$. The result, which we have obtained, is based on the classification of all non-equivalent indecomposable integral matrix representations of the group $S_3$, obtained by L. A. Nazarova and A. V. Roiter [**1**]. The following representations of the group $S_3$ over the ring $\mathbb{Z}$ of rational integers presents all indecomposable integral pairwise