Corollary 1. *Transforming matrices U and V from (2) have the following upper unitriangular form*

$$U = \begin{bmatrix} I & -Y \\ 0 & I \end{bmatrix}, V = \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix},$$

where matrices X and Y have the same triangular form as matrices A, B and C if and only if $(a_{ii}, b_{ii})|c_{ii}$ for all $i = 1, 2, \ldots, n$.

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Some notes on orthogonality

Iryna Fryz

A tuple of n-ary operations $f_1, \ldots, f_k$ $(n \geq 2, k \leq n)$ defined on a set Q $(m := |Q|)$ is called *orthogonal* [**1**], if for arbitrary $b_1, \ldots, b_k \in Q$ the system $\{f_i(x_1, \ldots, x_n) = b_i\}_{i=1}^k$ has exactly m^{n-k} solutions.

Let f be an n-ary operation on Q and

$$\delta := \{i_1, \ldots, i_k\} \subset \overline{1,n} := \{1, \ldots, n\}, \quad \{j_1, \ldots, j_{n-k}\} := \overline{1,n} \setminus \delta, \quad \bar{a} := (a_{j_1}, \ldots, a_{j_{n-k}}).$$

An operation $f_{(\bar{a},\delta)}$ which is defined by

$$f_{(\bar{a},\delta)}(x_{i_1}, \ldots, x_{i_k}) := f(y_1, \ldots, y_n),$$

where $y_i := \begin{cases} x_i, & \text{if } i \in \delta, \\ a_i, & \text{if } i \notin \delta, \end{cases}$ is called an $(\bar{a}, \delta)$*-retract* or a δ*-retract* of f. Operations $f_{1;(\bar{a}_1,\delta)}, \ldots, f_{k;(\bar{a}_k,\delta)}$ are called *similar* δ-retracts of n-ary operations $f_1, \ldots, f_k$, if $\bar{a}_1 = \cdots = \bar{a}_k$. A k-tuple of n-ary operations is called δ*-retractly orthogonal* [**4**], if all tuples of similar δ-retracts of these operations are orthogonal.

The notion of perpendicularity of the maximal type from [**3**] can be defined using the definition of retract orthogonality: n-ary operations g and h are called *perpendicular of the type* $(\iota, \iota; m)$, if they are δ-retractly orthogonal for all δ such that $|\delta| = 2$ i $m \in \delta$. The results from [**5**] imply the following statement.

Proposition 1. *If n-ary operations g and h are perpendicular of the type $(\iota, \iota; m)$, $m \in \overline{1,n}$, then they are δ-retractly orthogonal for all $\delta \subset \overline{1,n}$, where $|\delta| > 1$ and $m \in \delta$.*

The relationships between retract orthogonality and strong orthogonality was described by G.B. Belyavskaya and G.L. Mullen [**2**] and the relationships between retract orthogonality and orthogonality was studied in [**5**].

Proposition 2. *Let g and h be n-ary quasigroups. The following statements are equivalent:*

(1) g and h are strongly orthogonal;

(2) g and h are perpendicular of the type $(\iota,\iota;m)$ for all $m \in \overline{1,n}$;

(3) g and h are δ-retractly orthogonal for all $\delta \subset \overline{1,n}$;

(4) for an arbitrary $m \in \overline{1,n}$ operation $g \underset{m}{\oplus} h$ is invertible, where

$$(g \underset{m}{\oplus} h)(x_1,\dots,x_n) := g(x_1,\dots,x_{m-1},h(x_1,\dots,x_n),x_{m+1},\dots,x_n).$$

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Diagonal reduction of matrices over commutative semihereditary Bezout rings

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All rings considered will be commutative and have identity. Recently there has been some interest in the polynomial ring $R[x]$, where R is a von Neumann regular ring. Such a ring is a Bezout ring, semihereditary ring, and so Hermite ring. Thus, it is natural to ask whether or not $R[x]$ is an elementary divisor ring. This question is answered affirmative in [3]. It is an open problem whether or not every Bezout domain is an elementary divisor ring and more generally: whether or not every semihereditary Bezout ring is an elementary divisor ring.

We obtain a complete characterization of semihereditary elementary divisor ring through its homomorphic images.

Mc Adam S. and Swan R. G. studied comaximal factorization in commutative rings [2]. Following them, we give the following definitions.

Definition 1. A nonzero element a of a ring R is called inpseudo-irreducible if for any representation $a = b \cdot c$ we have $bR + cR = R$.

Definition 2. An element a of a ring R is called pseudo-irreducible if for any representation $a = b \cdot c$, where $b, c \notin U(R)$, we have $bR + cR \neq R$.

Other definitions can be found in the articles [1, 4].

Theorem 1. *Let R be a Bezout ring of stable range 2. A regular element $a \in R$ is inpseudo-irreducible iff R/aR is a von Neumann regular ring.*