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On some combinatorial identities involving the Horadam numbers

Taras Goy

In [5, 6], Horadam defined generalized Fibonacci numbers $\{w_n(a,b;p,q)\}$, or briefly $\{w_n\}$, which satisfy the second-order homogeneous linear recurrence relation

$$w_n = pw_{n-1} - qw_{n-2}, \quad n \geq 2, \tag{1}$$

where $w_0 = a$, $w_1 = b$ and a, b, p, q are integers.

This sequence generalizes many number sequences, such as Fibonacci, Lucas, Pell, Jacobsthal sequences, among others.

We study some families of Toeplitz-Hessenberg determinants the entries of which are Horadam numbers. These determinant formulas may also be rewritten as identities involving sums of products of the Horadam numbers and multinomial coefficients.

Let $\varepsilon = a^2q - abp + b^2$, $|s| = s_1 + s_2 + \cdots + s_n$, $\sigma_n = s_1 + 2s_2 + \cdots + ns_n$, and $p_n(s) = \frac{(s_1+\cdots+s_n)!}{s_1!\cdots s_n!}$ denotes the multinomial coefficient.

Theorem 1. *For all $n \geq 2$, the following formulas fold*

$$\sum_{\sigma_n = n} (-1)^{|s|} p_n(s) \left(\frac{w_1}{w_0}\right)^{s_1} \left(\frac{w_2}{w_0}\right)^{s_2} \cdots \left(\frac{w_n}{w_0}\right)^{s_n} = \frac{\varepsilon(ap-b)^{n-2}}{a^n},$$

$$\sum_{\sigma_n = n} (-1)^{|s|} p_n(s) \left(\frac{w_2}{w_0}\right)^{s_1} \left(\frac{w_4}{w_0}\right)^{s_2} \cdots \left(\frac{w_{2n}}{w_0}\right)^{s_n} = \frac{\varepsilon p^2(ap^2-aq+bp)^{n-2}}{a^n},$$

$$\sum_{\sigma_n = n} (-1)^{|s|} p_n(s) \left(\frac{w_2}{w_1}\right)^{s_1} \left(\frac{w_3}{w_1}\right)^{s_2} \cdots \left(\frac{w_{n+1}}{w_1}\right)^{s_n} = \frac{\varepsilon a^{n-2}q^{n-1}}{b^n},$$

$$\sum_{\sigma_n = n} (-1)^{|s|} p_n(s) \left(\frac{w_3}{w_1}\right)^{s_1} \left(\frac{w_5}{w_1}\right)^{s_2} \cdots \left(\frac{w_{2n+1}}{w_1}\right)^{s_n} = \frac{\varepsilon q^{n-1}p^2(ap-b)^{n-2}}{b^n},$$

where the summation is over integers $s_i \geq 0$ satisfying $s_1 + 2s_2 + \cdots + ns_n = n$.

These identities generalize some identities which we have obtained in [1–4].

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On σ-nilpotency of finite groups

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All considered groups are finite and G always denotes a finite group. The symbol $\pi(G)$ denotes the set of all primes dividing the order of G. Two groups A and B are called *isoordic* if $|A| = |B|$.

Let σ be some partition of the set of all primes $\mathbb{P}$, that is, $\sigma = \{\sigma_i | i \in I\}$, where $\mathbb{P} = \bigcup_{i\in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$, and we put, following [**5**], $\sigma(G) = \{\sigma_i | \sigma_i \cap \pi(G) \neq \emptyset\}$. G is said to be: *σ-primary* [**5**] if G is a σ_i-group for some i; *σ-decomposable* (Shemetkov [**4**]) or *σ-nilpotent* (Guo and Skiba [**1**]) if $G = G_1 \times \cdots \times G_n$ for some σ-primary groups $G_1, \ldots, G_n$.

A subgroup A of G is called *σ-subnormal* in G [**5**] if it is *$\mathfrak{N}_\sigma$-subnormal in G* in the sense of Kegel [**2**], that is, there is a subgroup chain

$$A = A_0 \leq A_1 \leq \cdots \leq A_n = G$$

such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is σ-primary for all $i = 1, \ldots, n$. We use $i_\sigma(G)$ to denote the *number of classes of isoordic non-σ-subnormal subgroups* of G.

We study the structure of G depending on the invariant $i_\sigma(G)$. In particular, we obtained the conditions of σ-nilpotency of G with restrictions on $i_\sigma(G)$. For example, the following theorem was proved.

Theorem. [**3**, Theorem 1.7] *If $i_\sigma(G) \leq |\sigma(G)| - 2$, then G is σ-nilpotent.*

Note that Theorem is a corollary of the more general result [**3**, Theorem 1.2].

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