Can one hear the shape of a group?

Rostislav I. Grigorchuk, Artem Dudko

With a finitely generated group G generated by $S = \{s_1, \ldots, s_m\}$ one can associate its Cayley graph $\Gamma = \Gamma(G, S) = (V, E)$ where the set of vertices $V = G$ and the set of edges $E = \{(g, sg) : g \in G, s \in S\}$. The question, raised independently by A. Valette in 1993 and K. Fujiwara in 2016, is "Can one hear the shape of a group?". It is the analogue of the famous question raised by M. Kac in 1966 "Can one hear the shape of a drum?"

The group analogue of Kac's question asks if the spectrum of the Cayley graph of a group determines it up to isometry. By a spectrum of graph $\Gamma = (V, E)$ we mean spectrum of Markov operator M acting in $l^2(V)$ according to

$$Mf(v) = \frac{1}{deg(v)} \sum_{w} f(w),$$

where summation is taken over all neighbor vertices w of v and $deg(v)$ denotes the degree of the vertex v. In the case of d-regular graph (when $deg(v) = d$ for all $v \in V$) instead of M one can consider the operator $\Delta = I - M$ (I is the identity operator) which is called discrete Laplace operator. There is no difference to study spectral properties of Δ or M but we prefer to deal with operator M. In the case of the d-regular graph the matrix of M is equal to $\frac{1}{d}A$ where A is the adjacency matrix of Γ. Cayley graphs, as well as their generalizations Schreier graphs, are d-regular with $d = 2|S|$.

In my talk I will explain that the answer to the Valette-Fujiwara question is negative in a very strong sense.

Let $\mathcal{G} =< a, b, c, d >$ be a group of intermediate growth between polynomial and exponential constructed by the speaker in 1980. Let $\mathcal{G}_\omega, \omega \in \Omega = \{0, 1, 2\}^{\mathbb{N}}$ be an uncountable family of groups of intermediate growth constructed by the speaker in 1983. The group $\mathcal{G}$ belongs to this family and corresponds to the periodic sequence $\omega = (012)^\infty$. Additionally to the property to have intermediate growth, most of the groups from the family are infinite torsion (hence of Burnside type) groups, amenable but not elementary amenable, branch, just infinite, etc. The canonical generating set $S_\omega = \{a, b_\omega, c_\omega, d_\omega\}$ satisfies the standard relations: generators are involutions and $b_\omega, c_\omega, d_\omega$ commute, but the groups are not finitely presentable. Moreover, as proven by M.G. Benli, P. de la Harpe and speaker at [1] any amenable cover $\tilde{\mathcal{G}}$ of any $\mathcal{G}_\omega$ is infinitely presented, so these groups do not have amenable finitely presented cover.

Theorem 1 (A. Dudko, R. Grigorchuk). *For any $\omega \in \Omega$ the spectrum $sp(M_\omega)$ of the corresponding Markov operator for $\Gamma(\mathcal{G}_\omega, S_\omega)$ is equal to the union Λ of two intervals* $[-1/2, 0] \cup [1/2, 1]$.

Among groups $\mathcal{G}_\omega, \omega \in \Omega$ there are uncountably many pairwise not quasi-isometric because of the different rate of growth. Therefore the above theorem show that there are uncountable many not only not pairwise isometric but even not pairwise quasi-isometric groups with the same spectrum. Thus the spectrum of Cayley graphs does not provide enough information to reconstruct it.

Theorem 2 (A. Dudko, R. Grigorchuk). *For each not virtually constant sequence $\omega \in \Omega$ there are uncountably many pairwise non isomorphic amenable groups $\bar{G}$ generated by a set $\bar{S} = \{\bar{a}, \bar{b}, \bar{c}, \bar{d}\}$ consisting of four involutions with $\bar{b}, \bar{c}, \bar{d}$ commuting, such that the map $\phi : \bar{a} \to a, \bar{b} \to b_\omega, \bar{c} \to c_\omega, \bar{d} \to d_\omega$ extends to surjective homomorphism $\bar{G} \to \mathcal{G}_\omega$, and the spectrum of $\Gamma(\bar{G}, \bar{S})$ is equal to Λ.*

This theorem again show that the spectrum can not determine the structure of the Cayley graphs. The proof of the above results, in particular, uses the Hulanicki criterion of amenability of groups in terms of the weak containment of unitary representations, as well as its weak analogue for spectra of covering graphs obtained by A. Dudko and speaker.

Define a **spectral measure** by $\mu(B) = \langle E(B)\delta_e, \delta_e\rangle$ where $B \subset \mathbb{R}$ Borel subsets, $\{E(B)\}$ spectral projections associated with M, and δ_e is a delta function at identity $e \in G$.

Problem 1. Is it correct that the spectral measure μ associated with Markov operator on the Cayley graph of a finitely generated group determines it up to isometry?

μ determines the spectrum of M, probabilities $P^{(n)}_{e,e}$ of return, the Ihara zeta function, $\ldots$. Perhaps the answer could be affirmative.

The stated results are presented in [2].

References

1. M. G. Benli, P. de la Harpe, R.Grigorchuk, *Amenable groups without finitely presented amenable covers,*, Bull. Math. Sci. 3 (2013), no. 1, 73-131.
2. A. Dudko, R. Grigorchuk, *On the question "Can one hear the shape of a group and weak Hulanicki theorem for graphs"*, arXiv https://arxiv.org/pdf/1809.04008.

Contact information

Rostislav I. Grigorchuk
Department of Mathematics, Texas A&M University, College Station, USA
Email address: grigorch@math.tamu.edu
URL: https://www.math.tamu.edu/~grigorch/

Artem Dudko
Institute of Mathematics, Polish Academy of Science, Warsaw, Poland
Email address: adudko@impan.pl
URL: https://www.impan.pl/pl/sites/adudko/home

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On inverse submonoids of the monoid of almost monotone injective co-finite partial selfmaps of positive integers

Oleg Gutik

We follow the terminology of [**1, 3, 4, 6, 7**].

The monoid $\mathscr{I}_{\infty}^{\,\Rsh\!\!\nearrow}(\mathbb{N})$ of almost monotone injective co-finite partial selfmaps of positive integers and the monoid $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ of monotone injective co-finite partial selfmaps of positive integers studied in [**2**] and [**5**].

In our report we discuss on submonoids of the monoid $\mathscr{I}_{\infty}^{\,\Rsh\!\!\nearrow}(\mathbb{N})$. We give a small survey and announce new results on this topic.

Let $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ be a submonoid of $\mathscr{I}_{\infty}^{\,\Rsh\!\!\nearrow}(\mathbb{N})$ which consists of cofinite monotone partial bijections of $\mathbb{N}$ and $\mathscr{C}_{\mathbb{N}}$ be a subsemigroup $\mathscr{I}_{\infty}^{\,\Rsh\!\!\nearrow}(\mathbb{N})$ which is generated by the partial shift $n \mapsto n+1$ and its inverse partial map. We show that every automorphism of a full inverse subsemigroup of $\mathscr{I}_{\infty}^{\nearrow}(\mathbb{N})$ which contains the semigroup $\mathscr{C}_{\mathbb{N}}$ is the identity map. We construct a submonoid $\mathbf{I}\mathbb{N}_{\infty}^{[\underline{1}]}$ of $\mathscr{I}_{\infty}^{\,\Rsh\!\!\nearrow}(\mathbb{N})$ with the following property: if S is an inverse submonoid of $\mathscr{I}_{\infty}^{\,\Rsh\!\!\nearrow}(\mathbb{N})$ such that S contains $\mathbf{I}\mathbb{N}_{\infty}^{[\underline{1}]}$ as a submonoid, then every non-identity congruence $\mathfrak{C}$ on S is a group congruence. We show that if S is an inverse submonoid of $\mathscr{I}_{\infty}^{\,\Rsh\!\!\nearrow}(\mathbb{N})$ such that S contains $\mathscr{C}_{\mathbb{N}}$ as a submonoid