The number of integer polynomials whose discriminants are divided by a large prime power

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Let

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0,\ a_j \in \mathbb{Z},\ 0 \le j \le n, \tag{1}$$

is an integer polynomial of degree $\deg P = n$ (this means $a_n \neq 0$), the height $H = H(P) = \max_{0\le j\le n}|a_j| \le Q$ and roots α_1, α_2,...,α_n.

Then the discriminant $D(P)$ of the polynomial (1) is equal to

$$D(P) = a_n^{2n-2} \prod_{1\le i<j\le n} (\alpha_i - \alpha_j)^2. \tag{2}$$

The expression (2) is often taken as a definition of the discriminant.

For $1 \le v \le n-1$ and a natural number $Q > 1$ introduce a class $\mathcal{P}_n(Q, v)$ of polynomials

$$\mathcal{P}_n(Q, v) = \{P(x) \mid \deg P \le n,\ 1 \le D(P) < Q^{2n-2-2v}\}. \tag{3}$$

Denote $\#\mathcal{P}_n(Q, v)$ the number of elements of the finite set $\mathcal{P}_n(Q, v)$. In [**1**] was proven that

$$\#\mathcal{P}_n(Q, v) > c_1(n)Q^{n+1-\frac{n+2}{n}v}. \tag{4}$$

Estimates from above for the $\#\mathcal{P}_n(Q, v)$ were received in [**2**] for $n = 2$ and $n = 3$.

Let $|a|_p$ – p-adic norm of a natural number a. Similarly to (3) define a class of polynomials

$$\mathcal{P}^*_n(Q, v) = \{P(w) \mid \deg P \le n,\ |D(P)|_p < Q^{-2v}\}. \tag{5}$$

Theorem 1. *Let $2 \le n \le 4$ and $\varepsilon > 0$. Then*

$$\#\mathcal{P}^*_n(Q, v) < Q^{n+1-\frac{n+2}{n}v+\varepsilon}. \tag{6}$$

References

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