On injectors and Fischer subgroups of a finite π-soluble group

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Throughout this paper all groups are finite. The notations and terminologies are standard as in [**1**].

A nonempty set $\mathcal{F}$ of subgroups of G [**3**] is called *a Fitting set* of G if the following three conditions are satisfied: (i) If $T \trianglelefteq S \in \mathcal{F}$, then $T \in \mathcal{F}$; (ii) If $S, T \in \mathcal{F}$ and $S, T \trianglelefteq ST$, then $ST \in \mathcal{F}$; (iii) If $S \in \mathcal{F}$ and $x \in G$, then $S^x \in \mathcal{F}$.

Let $\mathbb{P}$ be the set of all primes and let $\emptyset \neq \pi \subseteq \mathbb{P}$, and $\pi' = \mathbb{P} \backslash \pi$. A Fitting set $\mathcal{F}$ of a group G is said to be *π-saturated* if $\mathcal{F} = \{H \leq G : H/H_{\mathcal{F}} \in \mathfrak{E}_{\pi'}\}$, where $\mathfrak{E}_{\pi'}$ is the class of all π'-groups. Let $\mathcal{F}$ be a Fitting set of G. An $\mathcal{F}$-subgroup F of G is said to be a *Fischer $\mathcal{F}$-subgroup* of a group G if F contains every $\mathcal{F}$-subgroup of G which is normalized by F.

If $\mathcal{F}$ is a Fitting set of a group G, then a subgroup V of G is said to be

(1) *$\mathcal{F}$-maximal* in G, if $V \in \mathcal{F}$ and $U = V$ provided that $V \leq U \leq G$ and $U \in \mathcal{F}$.

(2) an *$\mathcal{F}$-injector* of G, if $V \cap N$ is an $\mathcal{F}$-maximal subgroup of N for every subnormal subgroup N of G.

It is easy to see that in a soluble group G every $\mathcal{F}$-injector of G is a Fischer $\mathcal{F}$-subgroup of G. However, there exists Fitting sets $\mathcal{F}$ of G and a soluble groups G such that a Fischer $\mathcal{F}$-subgroup of G is not $\mathcal{F}$-injector and Fischer $\mathcal{F}$-subgroups are not conjugate (see [**1**, VIII. (4.9)]).

A Fitting set $\mathcal{F}$ of G is a *Fischer π-set* of G if $H \in \mathcal{F}$ whenever $K \trianglelefteq L \in \mathcal{F}$ and H/K is a p-subgroup of L/K for some prime $p \in \pi$. If $\pi = \mathbb{P}$, then the Fischer π-set of G is a Fischer set of G (see [**1**, p. 554]).

It is proved

Theorem 1. *Let $\mathcal{F}$ be a π-saturated Fischer π-set of a π-soluble group G. Then a subgroup V of G is an $\mathcal{F}$-injector of G if and only if V is a Fischer $\mathcal{F}$-subgroup of G containing a Hall π'-subgroup of G.*

Corollary 1. *Let $\mathcal{F}$ be a π-saturated Fischer π-set of a π-soluble group G. Then the Fischer $\mathcal{F}$-subgroups containing a Hall π'-subgroup of G are conjugate in G.*

Corollary 2 (Fischer [**2**]). *Let $\mathfrak{F}$ be a Fischer class of soluble groups. Then every soluble group G has a unique conjugate class of Fischer $\mathfrak{F}$-subgroups.*

References

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