Closure operators in Morita contexts: mappings and their properties

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Closure operator of a module category R-Mod is a function C, which associates to every submodule $N \subseteq M$, where $M \in R$-Mod, a submodule $C_M(N) \subseteq M$, which satisfies the conditions of *extension, monotony* and *continuity* ([**2, 3**]) Denote by $\mathbb{CO}(R)$ the class of all closure operators of R-Mod.

Let $(R,\, {}_RU_S,\, {}_SV_R,\, S)$ be an arbitrary *Morita context* with the morphisms $(,) : U \otimes_S V \to R$ and $[,] : V \otimes_R U \to S$ ([**1**]). We consider the functors $R\text{-Mod} \underset{H^V = Hom_S(V,\text{-})}{\overset{H^U = Hom_R(U,\text{-})}{\rightleftarrows}} S\text{-Mod}$ with the associated natural transformations $\varphi : \mathbb{1}_{R-\mathrm{Mod}} \to H^V H^U$ and $\psi : \mathbb{1}_{S-\mathrm{Mod}} \to H^U H^V$.

The purpose of this study is to establish the relation between the classes of closure operators $\mathbb{CO}(R)$ and $\mathbb{CO}(S)$ determined by the functors H^U and H^V for the given Morita context $(R,\, {}_RU_S,\, {}_SV_R,\, S)$. For that two mappings are constructed $\mathbb{CO}(R) \underset{(-)^*}{\overset{(-)^*}{\rightleftarrows}} \mathbb{CO}(S)$ between the classes of closure operators. The transition $C \rightsquigarrow C^*$, where $C \in \mathbb{CO}(R)$, is defined by the rule: $(C)^*_Y(N) \overset{\text{def}}{=\!=} \mathrm{Ker}\left[\psi_Y \cdot H^U(\pi^n_C)\right]$, where $n : N \overset{\subseteq}{\longrightarrow} Y$ is an inclusion of S-Mod and $\pi^n_C : H^V(Y) \longrightarrow H^V(Y)/C_{H^V(Y)}\big(Im\, H^V(n)\big)$ is a natural epimorphism. Similarly, $D \rightsquigarrow D^*$ is defined for $D \in \mathbb{CO}(S)$.

Some important properties of "star" mappings are proved. In particular:

1) the "star" mappings are monotone, i.e. $C_1 \le C_2 \Rightarrow C_1^* \le C_2^*$ and $D_1 \le D_2 \Rightarrow D_1^* \le D_2^*$;
2) $C \le C^{**}$ for every $C \in \mathbb{CO}(R)$, $D \le D^{**}$ for every $D \in \mathbb{CO}(S)$;
3) $\big(\bigwedge_{\alpha \in \mathfrak{A}} C_\alpha\big)^* = \bigwedge_{\alpha \in \mathfrak{A}} C^*_\alpha$ for every family $\{C_\alpha \mid \alpha \in \mathfrak{A}\} \subseteq \mathbb{CO}(R)$;
4) $\big(\bigwedge_{\alpha \in \mathfrak{A}} D_\alpha\big)^* = \bigwedge_{\alpha \in \mathfrak{A}} D^*_\alpha$ for every family $\{D_\alpha \mid \alpha \in \mathfrak{A}\} \subseteq \mathbb{CO}(S)$.

References

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