Lie algebras of derivations with large abelian ideals

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Let $\mathbb{K}$ be a field of characteristic zero, $A = \mathbb{K}[x_1, ..., x_n]$ the polynomial ring and $R = \mathbb{K}(x_1, \ldots, x_n)$ the field of rational functions in n variables. The Lie algebra $\widetilde{W}_n(\mathbb{K}) := Der_{\mathbb{K}}R$ of all $\mathbb{K}$-derivation of R is of great interest because elements of $\widetilde{W}_n(\mathbb{K})$ (which are of the form $D = f_1\frac{\partial}{\partial x_1} + \cdots + f_n\frac{\partial}{\partial x_n}, f_i \in R$) can be considered as vector fields on $\mathbb{K}^n$ with rational coefficients $f_1, \ldots, f_n \in R$. Lie algebras of vector fields were studied by many authors (see for example [**2**], [**1**] and others).

Since $\widetilde{W}_n(\mathbb{K})$ is a vector space of dimension n over R one can define the rank $\mathrm{rk}_R L$ over R for any subalgebra $L \subseteq \widetilde{W}_n(\mathbb{K})$ by the rule: $\mathrm{rk}_R L := \dim_R RL$. We study subalgebras $L \subseteq \widetilde{W}_n(\mathbb{K})$ of rank m over R which have an abelian ideal I of the same rank m over R. A natural basis over F, the field of constants for L in R, for such Lie algebras is built.

Theorem 1. *Let L be a subalgebra of rank m over R of the Lie algebra $\widetilde{W}_n(\mathbb{K})$ with an abelian ideal $I \leq L$ of the same rank m over R. Then there exist a basis $D_1, \ldots, D_m$ of the Lie algebra FL over F and elements $a_1, \ldots, a_k \in R, k \geq 1$ such that $D_i(a_j) = \delta_{ij}, i = 1, \ldots, m, j = 1, \ldots, k$. Every element $D \in FL$ can be written in the form $D = \sum_{i=1}^{m} f_i(a_i, \ldots, a_k)D_i$ for some polynomials $f \in \mathbb{K}[t_1, \ldots, t_k], degf_i \leq 1$.*

Using properties of this basic one can prove that the Lie algebra FL over the field F can be isomorphically embedded into the general affine Lie algebra $\mathrm{aff}_m(F)$. This result can be used to study solvable finite dimensional subalgebras $L \subseteq \widetilde{W}_n(\mathbb{K})$. Note that in cases $n = 2$ and $n = 3$ this theorem was proved in [**3**] and [**4**] respectively.

References

1. *A. González-López A., N. Kamran N., P.J. Olver, Lie algebras of differential operators in two complex variables,* Amer. J. Math. – 1992. – **114**. – P.1163–1185.
2. S. Lie, *Theorie der Transformationsgruppen*, v.3, Leipzig, 1893.
3. *Ie.Makedonskyi, A.Petravchuk, On nilpotent and solvable Lie algebras of derivations,* Journal of Algebra. – 2014. – **401**. – P.245–257.
4. Ie.Chapovskyi, D.Efimov, A.Petravchuk, *Solvable Lie algebras of derivations of polynomial rings in three variables*, Applied problems of Mechanics and Mathematics. - Lviv, 2018, issue. 16. - P.7-13.

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On eigenvalues of random partial wreath product

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Partial wreath n-th power of symmetric inverse semigroup $\mathcal{I}_d$ is a semigroup defined recursively by

$$\mathcal{P}_n = (\mathcal{P}_{n-1}) \wr_p \mathcal{I}_d = \{(f,a) | a \in \mathcal{I}_d, f : \mathrm{dom}(a) \to \mathcal{P}_{n-1}\}, n \geq 2,$$

with composition

$$(f,a) \cdot (g,b) = (fg^a, ab),$$

and $\mathcal{P}_1 = \mathcal{I}_d$.

To a randomly chosen element $x \in \mathcal{P}_n$, we assign the matrix

$$A_x = \left(1_{\{x(v_i^n)=v_j^n\}}\right)_{i,j=1}^{d^n}.$$

In other words, (i,j)-th entry of A_x is equal to 1 if transformation x maps i to j, and 0 otherwise.

Let $\chi_x(\lambda)$ be the characteristic polynomial of A_x and $\lambda_1, \ldots, \lambda_{d^n}$ be its roots respecting multiplicity.

Denote by

$$\Xi_n = \frac{1}{d^n} \sum_{k=1}^{d^n} \delta_{\lambda_k}$$

a uniform distribution on eigenvalues of A_x.

Let $x \in \mathcal{P}_n$ let $\eta_n(x) = \Xi_n(0)$ be a fraction of zero eigenvalues A_x and $\xi_n(x) = 1 - \eta_n(x)$ be a fraction of nonzero eigenvalues. Then we have

$$\eta_n(x) \xrightarrow{\mathbb{P}} 1, n \to \infty,$$

or, equivalently,

$$\xi_n(x) \xrightarrow{\mathbb{P}} 0, n \to \infty.$$

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