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Classification of quasigroup functional equations and identities of minimal length

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A groupoid $(Q;\cdot)$ is called a *quasigroup*, if for all a, $b \in Q$ every of the equations $x \cdot a = b$ and $a \cdot y = b$ has a unique solution. A *σ-parastrophe* $(Q;\overset{\sigma}{\cdot})$ of $(Q;\cdot)$ is defined by

$$x_{1\sigma} \overset{\sigma}{\cdot} x_{2\sigma} = x_{3\sigma} \iff x_1 \cdot x_2 = x_3, \quad \sigma \in S_3.$$

A *σ-parastrophe of a class* of quasigroups $\mathfrak{A}$ is called a class ${}^{\sigma}\mathfrak{A}$, which consists of all σ-parastrophes of quasigroups from $\mathfrak{A}$ [**4**].

Two identities are called:

— *equivalent*, if they determine the same variety;
— *parastrophically equivalent*, if they determine parastrophic varieties.

Evidently that every equinelent identity are parastrophically equivalent, but the inverse is not valid.

A *parastrophic symmetry group* of a variety $\mathfrak{A}$ is $\mathrm{Ps}(\mathfrak{A}) := \{\sigma \mid {}^{\sigma}\mathfrak{A} = \mathfrak{A}\}$ and it is subgroup of the group S_3. A variety is called:

— totally-symmetric, if $\mathrm{Ps}(\mathfrak{A}) = S_3$;
— semisymmetric, if $\mathrm{Ps}(\mathfrak{A}) = A_3$;
— one-sided-symmetric, if $|\mathrm{Ps}(\mathfrak{A})| = 2$;
— asymmetric, if $|\mathrm{Ps}(\mathfrak{A})| = 1$.

A truss of varieties is called the set of all pairwise parastrophic varieties. A truss of varieties is uniquely defined by an identity which describes one of varieties from the given truss. A truss will be called: *totally-symmetric*, if it has 1 variety; *semisymmetric*, if it has two varieties; *one-sided-symmetric*, if it has three varieties; *asymmetric*, if it has six varieties.

A *length* of an identity is defined as the number of all functional symbols (not necessary different) appearing in it. Any quasigroup identity of length 1 is equivalent to the identity of idempotency.

Theorem 1. *An arbitrary quasigroup identity of length 2 is equivalent to exactly one of the following 14 identities and is parastrophically-equivalent to exactly one of the 6 identities having different numbers:*

$$\begin{array}{llll}
1)\ x = x, & 2)\ xy \cdot x = y, & & \\
3)\ xy = yx, & 4)\ x^2 = y^2, & 5)\ x^2 \cdot y = y, & 6)\ x^2 \cdot x = x, \\
{}^{\ell}3)\ x \cdot xy = y, & {}^{\ell}4)\ (x \overset{\ell}{\cdot} x)y = y, & {}^{s}5)\ x \cdot y^2 = x, & {}^{s}6)\ x \cdot x^2 = x, \\
{}^{r}3)\ xy \cdot y = x, & {}^{r}4)\ x(y \overset{r}{\cdot} y) = x, & {}^{\ell}5)\ x(y \overset{\ell}{\cdot} y) = x, & {}^{\ell}6)\ x(x \overset{\ell}{\cdot} x) = x.
\end{array}$$

Corollary 1. *All quasigroup identities of length 2 determine 14 different varieties distributing in 6 trusses according to the law of parastrophic symmetry. The trusses* 1), 2) *are totally-symmetric and the trusses* 3), 4), 5), 6) *are one-sided-symmetric.*

Remark that in Theorem 1, the identity 1) determines truss of all quasigroups; 2) determines truss of all semisymmetric quasigroups; 3) is the truss of all commutative quasigroups; 4) is the truss of one-sided loops; 5) is a truss of two-sided loops; 6) is a truss of quasigroups defined by the identity $x^2 \cdot x = x$.

Theorem 2. *Any quasigroup identity of length 3 is equivalent to exactly one of the following 74 identities and is parastrophically-equivalent to exactly one of the 20 identities having different numbers:*

1)	$x=y$	2)	$x^2=x$	3)	$x^2=x \wedge yx\cdot y=x$
4)	$x^2=x \wedge xy=yx$,	$^{\ell}$4)	$x^2=x \wedge x\cdot xy=y$,	^{r}4)	$x^2=x \wedge xy\cdot y=x$,
5)	$x\cdot xy=yx$,	^{s}5)	$yx\cdot x=xy$,	$^{\ell}$5)	$x(y\cdot yx)=yx$,
^{r}5)	$(x\cdot xy)y=x$,	$^{s\ell}$5)	$y(yx\cdot x)=x$,	^{sr}5)	$(xy\cdot y)x=xy$,
6)	$xy\cdot x=y\cdot xy$,	$^{\ell}$6)	$y(x\cdot yx)=x$,	^{r}6)	$(xy\cdot x)y=x$,
7)	$yx\cdot xy=x$,	$^{\ell}$7)	$y(xy\cdot x)=x$,	^{r}7)	$(x\cdot yx)y=x$,
8)	$x(x\cdot xy)=y$,	^{s}8)	$(yx\cdot x)x=y$,	$^{\ell}$8)	$x(yx \overset{\ell}{\cdot} y)=yx$,
9)	$y(x\cdot xy)=x$,	^{s}9)	$(yx\cdot x)y=x$,	$^{\ell}$9)	$x(yx\cdot y)=yx$,
^{r}9)	$(x\cdot xy)x=y$,	$^{s\ell}$9)	$(xy\cdot x)x=y$,	^{sr}9)	$(x\cdot yx)y=yx$,
10)	$x^2\cdot xy=y$,	^{s}10)	$yx\cdot x^2=y$,	$^{\ell}$10)	$xy\cdot yx=yx$,
^{r}10)	$x\cdot(x\overset{r}{\cdot}x)y=y$,	$^{s\ell}$10)	$xy\cdot yx=xy$,	^{sr}10)	$y(x\overset{\ell}{\cdot}x)\cdot x=y$,
11)	$xy\cdot x^2=y$,	$^{\ell}$11)	$x(yx\cdot y)=yx\cdot y$,	^{r}11)	$x(y\cdot xy)=x$,
12)	$yx^2\cdot y=x$,	^{s}12)	$y\cdot x^2y=x$,	$^{\ell}$12)	$xy\cdot(x\overset{\ell}{\cdot}x)=y$,
^{r}12)	$x(yx\cdot y)=x$,	$^{s\ell}$12)	$(x\overset{r}{\cdot}x)\cdot yx=y$,	^{sr}12)	$(y\cdot xy)x=x$,
13)	$xy^2\cdot x=x$,	^{s}13)	$x\cdot y^2x=x$,	$^{\ell}$13)	$x^2(y\overset{\ell}{\cdot}y)=x$,
^{r}13)	$y\cdot x(x\overset{\ell}{\cdot}x)=y$,	$^{s\ell}$13)	$(y\overset{r}{\cdot}y)x^2=x$,	^{sr}13)	$(x\overset{r}{\cdot}x)x\cdot y=y$,
14)	$x^2x\cdot y=y$,	^{s}14)	$y\cdot xx^2=y$,	$^{\ell}$14)	$y^2x\cdot x=x$,
^{r}14)	$(x\overset{r}{\cdot}x)(y\overset{\ell}{\cdot}y)=x$,	$^{s\ell}$14)	$x\cdot xy^2=x$,	^{sr}14)	$(y\overset{r}{\cdot}y)(x\overset{\ell}{\cdot}x)=x$,
15)	$xx^2\cdot y=y$,	^{s}15)	$y\cdot x^2x=y$,	$^{\ell}$15)	$y^2(x\overset{\ell}{\cdot}x)=x$,
^{r}15)	$x\cdot x(y\overset{\ell}{\cdot}y)=x$,	$^{s\ell}$15)	$(x\overset{r}{\cdot}x)y^2=x$,	^{sr}15)	$(y\overset{r}{\cdot}y)x\cdot x=x$,
16)	$x^2\cdot x^2=x$,	$^{\ell}$16)	$x\cdot(x\overset{r}{\cdot}x)x=x$,	^{r}16)	$x(x\overset{\ell}{\cdot}x)\cdot x=x$,
17)	$x^2x\cdot x=x$,	^{s}17)	$x\cdot xx^2=x$,	^{r}17)	$(x\overset{r}{\cdot}x)(x\overset{\ell}{\cdot}x)=x$,
18)	$xx^2\cdot x=x$,	^{s}18)	$x\cdot x^2x=x$,	$^{\ell}$18)	$x^2(x\overset{\ell}{\cdot}x)=x$,
^{r}18)	$x\cdot x(x\overset{\ell}{\cdot}x)=x$,	$^{s\ell}$18)	$(x\overset{r}{\cdot}x)x^2=x$,	^{sr}18)	$(x\overset{r}{\cdot}x)x\cdot x=x$,
19)	$xy\cdot y=x\cdot xy$,	20)	$xy\cdot yx=x$.		

Corollary 2. *All quasigroup identities of length 3 determine 74 different varieties distributing in 20 trusses according to the law of parastrophic symmetry. Five trusses* 1), 2), 3), 19), 20) *are totally-symmetric; eight trusses* 5), 9), 10), 12), 13), 14), 15), 18) *are asymmetric; seven trusses* 4), 6), 7), 8), 11), 16), 17) *are one-sided-symmetric; therefore, semisymmetric trusses does not exist.*

Remark that in Theorem 2, the identity 1) determines the truss of all trivial quasigroups; 2) determines the truss of all idempotent quasigroups; 3) is the truss of all idempotently semisymmetric quasigroups; 4) is the truss of all idempotently commutative quasigroups; 10) is truss of IP-quasigroups with invertible element x^2; 11) is the truss of all CIP-quasigroups with invertible element x^2; 13) is the truss of left loops with identity $x^2e = x$; 14) is the truss of all left loops with identity $x^2 \cdot x = e$; 15) is the truss of all left loops with identity $x \cdot x^2 = e$, where e is neutral element of the loops.

The identities 5), ^{r}5), ^{sr}5), 6), ^{r}6), 7), ^{r}7), ^{s}8), ^{s}9), ^{r}9), $^{s\ell}$9), 19), 20) are found by T. Evans [**1**], studying parastrophic orthogonality. Description of minimal non-trivial identities 5), 6), 7), 8), 9), 19), 20) are received by V. D. Belousov [**2**]. Regardless of him, the identities 5), 6), ^{r}6), 7), ^{s}8), ^{r}9), 19), 20) are highlighted by F. Bennett [**3**]. The parastrophic identities 5), ^{s}5), $^{\ell}$5), ^{r}5),

${}^{s\ell}5)$, ${}^{sr}5)$ are described by Sh. Stein [**5**]. The identity 5) is known as I Stein's law, 6) is II Stein's law, 7) is III Stein's law, 19) is I Shröder's law, 20) is II Shröder's law. The identity 8) we call I Belousov's law and identity 9) we call II Belousov's law.

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On reducibility of uncancellable generalized functional equations

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An algebra $(Q; f, {}^{\ell}f, {}^{r}f)$ is called a *binary quasigroup* [**2**] if it satisfies the following identities:

$$f({}^{\ell}f(x;y);y) = x, \quad {}^{\ell}f(f(x;y);y) = x, \quad f(x;{}^{r}f(x;y)) = y, \quad {}^{r}f(x;f(x;y)) = y. \qquad (1)$$

We consider a generalized quadratic binary quasigroup functional equations. Under the *functional equation* [**1**] we mean the universally quantified equality of the two terms $\upsilon = \omega$, which consists of functional and individual variables, and has no individual or functional constants (for general definition see [**7**]), while the carrier is considered to be an arbitrary set.

Two functional equations are said to be *parastrophically primarily equivalent* [**5**]–[**7**], if one can obtain from the other for a finite number of following steps: 1) using quasigroup identities (1); 2) rearranging parts of the equation; 3) renaming the individual variables; 4) renaming the functional variables.

Functional equation is called:

- *generalized*, if all the functional variables are pairwise different [**4**];
- *quadratic*, if every individual variable has exactly two appearance [**3**];
- *balanced*, if every individual variable has an appearance exactly once in the left and right sides of the equation [**3**];
- *binary*, if all functional variables are binary operations [**2**];
- *quasigroup*, if it is assumed that each functional variable acquires the values in the set of quasigroup operations of an arbitrary carrier [**5**].

A quasigroup functional equation is called *reducible* [**7**], if it is equivalent to a system of equations, each of which has a smaller number of diferent individual variables. A quadratic