${}^{s\ell}5)$, ${}^{sr}5)$ are described by Sh. Stein [**5**]. The identity 5) is known as I Stein's law, 6) is II Stein's law, 7) is III Stein's law, 19) is I Shröder's law, 20) is II Shröder's law. The identity 8) we call I Belousov's law and identity 9) we call II Belousov's law.

References

1. Evans T. *Algebraic structures associated with Latin squares and orthogonal arrays*, Proc. Conf. Algebraic Aspects of Combinatorics, Congressus Numerantium **13** (1975), 31–52.
2. Belousov V. D. *Parastrophically orthogonal quasigroups*, Preprint Acad. nauk Moldova SSR, Izd. Stiintsa, Kishinev, (1983), p. 50 (in Russian).
3. Bennett F. E. *The spectra of a variety of quasigroups and related combinatorial designs*, Discrete Mathematics **77** (1–3) (1989), 29–50.
4. Sokhatsky F. M. *Parastrophic symmetry in quasigroup theory*, Visnyk Donetsk national university, Ser. A: natural sciences No. 1–2 (2016), 70–83.
5. Stein Sherman K. *On the foundations of quasigroups*, Trans. Amer. Math. Soc. **85** (1957), 228–256.

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On reducibility of uncancellable generalized functional equations

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An algebra $(Q; f, {}^{\ell}f, {}^{r}f)$ is called a *binary quasigroup* [**2**] if it satisfies the following identities:

$$f({}^{\ell}f(x;y);y) = x, \quad {}^{\ell}f(f(x;y);y) = x, \quad f(x;{}^{r}f(x;y)) = y, \quad {}^{r}f(x;f(x;y)) = y. \qquad (1)$$

We consider a generalized quadratic binary quasigroup functional equations. Under the *functional equation* [**1**] we mean the universally quantified equality of the two terms $\upsilon = \omega$, which consists of functional and individual variables, and has no individual or functional constants (for general definition see [**7**]), while the carrier is considered to be an arbitrary set.

Two functional equations are said to be *parastrophically primarily equivalent* [**5**]–[**7**], if one can obtain from the other for a finite number of following steps: 1) using quasigroup identities (1); 2) rearranging parts of the equation; 3) renaming the individual variables; 4) renaming the functional variables.

Functional equation is called:

- *generalized*, if all the functional variables are pairwise different [**4**];
- *quadratic*, if every individual variable has exactly two appearance [**3**];
- *balanced*, if every individual variable has an appearance exactly once in the left and right sides of the equation [**3**];
- *binary*, if all functional variables are binary operations [**2**];
- *quasigroup*, if it is assumed that each functional variable acquires the values in the set of quasigroup operations of an arbitrary carrier [**5**].

A quasigroup functional equation is called *reducible* [**7**], if it is equivalent to a system of equations, each of which has a smaller number of diferent individual variables. A quadratic functional equation is said to be *parastrophically reducible* if it is parastrophically equivalent to a reducible equation. A quadratic quasigroup functional equation is called *cancellable* if it has a self-sufficient sequence of subterms (a sequence of subterms of the equation is called *self-sufficient* if it contains all the appearance of its individual variables in the equation). A quadratic quasigroup functional equation is called *parastrophically cancellable*, if it is parastrophically equivalent to a cancellable equation.

Theorem. [**6**] *An arbitrary generalized quadratic cancellable functional equation is reducible. An arbitrary generalized quadratic parastrophically cancellable equation is parastrophically reducible.*

It was proved in [**4**] that among all generalized quadratic binary quasigroup functional equations in six individual variables there are 14 uncancellable equations. We reviewed each of these 14 equations and found that they are all reducible. We give examples of a reducibility of these equations.

Theorem 1. *A generalized uncancellable quadratic binary quasigroup functional equation in six individual variables*

$$F_1(F_2(F_3(F_4(F_5(x;y);z);u);v);w) = F_6(x;F_7(y;F_8(z;F_9(u;F_{10}(v;w)))))$$

is equivalent to the following system of equations:

$$\begin{cases} F_1(F_2(x;y);z) = \gamma F_2(x;\rho^{-1}F_{10}(y;z)),\\ \gamma F_2(F_3(x;y);\rho^{-1}z) = \delta F_3(x;\mu^{-1}F_9(y;z)),\\ \delta F_3(F_4(x;y);\mu^{-1}z) = \alpha F_4(x;\beta^{-1}F_8(y;z)),\\ \alpha F_4(F_5(x;y);\beta^{-1}z) = F_6(x;F_7(y;z)), \end{cases}$$

where $\alpha,\beta,\gamma,\delta,\mu,\rho$ are arbitrary substitutions of the carrier set.

Theorem 2. *A generalized uncancellable quadratic binary quasigroup functional equation in six individual variables*

$$F_1(F_2(x,F_3(y,F_4(z,u))),F_5(v,w)) = F_6(F_7(F_8(x,y),z),(F_9(F_{10}(u,v),w))$$

is equivalent to the following system of equations:

$$\begin{cases} F_1(F_4(x;y);z) = F_6(\rho^{-1}\delta x;\beta^{-1}F_1(\alpha y;z)),\\ F_7(F_8(x;y);z) = \rho^{-1}\delta F_2(x;F_3(y;\gamma^{-1}z)),\\ F_9(F_{10}(x;y);z) = \beta^{-1}F_1(\alpha x;F_5(y;z)), \end{cases}$$

where $\alpha,\beta,\gamma,\delta,\rho$ are arbitrary substitutions of the carrier set.

Theorem 3. *A generalized uncancellable quadratic binary quasigroup functional equation in six individual variables*

$$F_1(F_2(F_3(x,y),(F_4(z,u)),(F_5(v,w)) = F_6(F_7(x,u),F_8(F_9(y,v),F_{10}(z,w)))$$

is equivalent to the following system of equations:

$$\begin{cases} F_1(\nu^{-1}F_6(\gamma^{-1}\nu x;y);v) = F_6(\gamma^{-1}F_1(x;v);y),\\ F_2(F_3(x;y);F_4(z;u)) = \nu^{-1}F_6(\gamma^{-1}\nu F_2(\alpha x;\beta z);F_7(y;u)),\\ F_8(F_9(y;v);F_{10}(z;w)) = \gamma^{-1}F_1(F_2(\alpha y;\beta z);F_5(v;w)), \end{cases}$$

where α,β,γ,ν are arbitrary substitutions of the carrier set.

Lemma 1. *All 14 generalized uncancellable quadratic binary quasigroup functional equations in six individual variables are reducible.*

Theorem 4. *All generalized quadratic binary quasigroup functional equations in six individual variables are reducible.*

References

1. Aczél J. *Lectures on functional equations and their applications.* New York, London: Academic Press, 1966. 509 p.

2. Belousov V.D. *Fundamentals of the Theory of Quasigroups and Loops*(In Russian). M.: Nauka , 1967. 222 p.
3. Krapez A. *Stricly quadratic functional equations on quasigroups.* Publ. Inst. Math. (Belgrad), 1981. tome 29 (43). pp. 125–138.
4. Krapez A., Simic S.K., Tosic D.V. *Parastrophically uncancellable quasigroup equations.* Aequationes Mathematicae, 2010. pp. 261–280.
5. Sokhatsky Fedir M. *Classification of the functional equations on quasigroups* (In Ukrainian). Ukrainian Mathematical Journal, 2004. 56, no. 9. pp. 1259–1266.
6. Sokhatsky Fedir M. *Associates and decompositions of multiary operations.* – Doctoral thesis (In Ukrainian), Vinnytsia–Kyiv, 2006.
7. Sokhatsky Fedir M. *Parastrophic symmetry in quasigroup theory.* Bulletin of Donetsk National University. Series A (Natural Sciences), 2016. no. 1–2. pp. 72–85.

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A matrix representation of Fibonacci and Lucas polynomials

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The k-Fibonacci and k-Lucas polynomials [2] are the natural extension of the k-Fibonacci and k-Lucas numbers and many of their properties admit a straightforward proof. The Fibonacci sequence and the golden ratio have appeared in many fields of science including high energy physics, cryptography and coding [1, 5].

Definition 1. The Fibonacci polynomial $F_n(x)$ is defined recurrently relation

$$F_{n+1}(x) = xF_n(x) + F_{n-1}(x) \tag{1}$$

with $F_0(x) = 0$, $F_1(x) = 1$ for $n \geq 1$.

Fibonacci polynomials for negative subscripts are defined as $F_{-n}(x) = (-1)^{n+1}F_n(x)$ for $n \geq 1$.

Definition 2. The Lucas polynomial $L_n(x)$ is defined by the relation

$$L_{n+1}(x) = xL_n(x) + L_{n-1}(x) \tag{2}$$

with $L_0(x) = 2$, $L_1(x) = x$ for $n \geq 1$ and $L_n(x) = F_{n+1}(x) + F_{n-1}(x)$ for $n \in \mathbb{Z}$.

If $x = 1$, the classic Fibonacci and Lucas sequences are obtained from (1), (2) [3-5].

Lemma 1. *If X is a square matrix with $X^2 = xX + I$, then $X^n = F_n(x)X + F_{n-1}(x)I$ for all $n \in \mathbb{Z}$.*