Leibniz algebras with the specifiec types of subalgebras

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Let L be an algebra over a field F with the binary operations $+$ and $[,]$. Then L is called a *Leibniz algebra* (more precisely a *left Leibniz algebra*), if it satisfies the (left) Leibniz identity: $[[a,[b,c]] = [[a,b],c] + [b,[a,c]]$ for all $a, b, c \in L$.

Leibniz algebra appeared first in the papers of A. M. Bloh [**1**], in which he called them D-algebras. Real interest in Leibniz algebras arose only after two decades thanks to the work of J. L. Loday [**2**].

A subspace A of a Leibniz algebra L is called a *subalgebra*, if $[x,y] \in A$ for all elements $x,\ y \in A$. A subalgebra A is called a *left* (respectively *right*) ideal of L, if $[y,x] \in A$ (respectively $[x,y] \in A$) for every $x \in A$, $y \in L$. In other words, if A is a left (respectively right) ideal, then $[L,A] \le A$ (respectively $[A,L] \le A$). The *center* $\zeta(L)$ of L is defined by the rule: $\zeta(L) = \{x \in L | [x,y] = 0 = [y,x]$ for each element $y \in L\}$. A subalgebra A of L is called an *contraideal* of L, if $A^L = L$. A Leibniz algebra L is called *extraspecial*, if $[L,L] = \zeta(L)$ has dimension 1.

Theorem 1. *Let L be a soluble Leibniz algebra, whose subalgebras are either ideals or contraideals. Then L is an algebra of one of following types:*

(i) *L is abelian;*

(ii) *$L = E \oplus Z$, where E is an extraspecial subalgebra such that $[e,e] \neq 0$ for each element $e \notin \zeta(E)$ and $Z \le \zeta(L)$;*

(iii) *$L = D \oplus Fb$, where $[y,y] = 0 = [b,b]$, $[b,y] = y = -[y,b]$ for every $y \in D$, in particular, L is a Lie algebra;*

(iv) *$L = D \oplus Fb$, where $[y,y] = [y,b] = 0 = [b,b]$, $[b,y] = y$ for every $y \in D$, in particular, $D = [L,L] = \mathbf{Leib}(L)$;*

(v) *$L = B \oplus A$, where $A = Fa_1 \oplus Fc_1$, $[a_1,a_1] = c_1, [c_1,a_1] = 0$, $[a_1,c_1] = c_1$ and $[b,b] = [b,a_1] = [b,c_1] = [c_1,b] = 0$, $[a_1,b] = b$ for every $b \in B$, in particular, $B \oplus Fc_1 = [L,L] = \mathbf{Leib}(L)$.*

(vi) *$\mathbf{char}(F) = 2$, $L = D \oplus Fa$, where D has a basis $\{z, b_\lambda | \lambda \in \Lambda\}$ such that $[a,a] = \alpha z$, $[a,b_\lambda] = b_\lambda = [b_\lambda,a]$, $[a,z] = [z,a] = 0$,$[z,b_\lambda] = [b_\lambda,z] = 0$ and $0 \neq [b_\lambda,b_\lambda] \in Fz$, $\lambda \in \Lambda$, $[b_\lambda,b_\mu] = 0$ for all $\lambda,\mu \in \Lambda$, $\lambda \neq \mu$, in particular, $D = [L,L]$, $Fz = \mathbf{Leib}(L)$.*

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Cramer's rules for Sylvester-type quaternion matrix equations

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Consider the two-sided generalized Sylvester matrix equation

$$\mathbf{AXB} + \mathbf{CYD} = \mathbf{E} \tag{1}$$

over the quaternion skew field $\mathbb{H}$. The Sylvester matrix equation has far reaching applications in different fields (see, e.g., [**1**]). Its solving is based on generalized inverses which are important tools in solving of matrix equations. Let for $\mathbf{A} \in \mathbb{H}^{m\times n}$, $\mathbf{A}^\dagger$ mean its Moore-Penrose generalized inverse, i.e. the exclusive matrix $\mathbf{X} \in \mathbb{H}^{n\times m}$ satisfying $\mathbf{AXA} = \mathbf{A}$, $\mathbf{XAX} = \mathbf{X}$, $(\mathbf{AX})^* = \mathbf{AX}$, $(\mathbf{XA})^* = \mathbf{XA}$. Furthermore, let $\mathbf{L}_A = \mathbf{I} - \mathbf{A}^\dagger\mathbf{A}$ and $\mathbf{R}_A = \mathbf{I} - \mathbf{AA}^\dagger$ be a couple of projectors induced by $\mathbf{A}$. In [**2**] the solvability conditions to Eq. (1) was obtained and its general solution was expressed in terms of generalized inverses as follows:

$$\mathbf{X} = \mathbf{A}^\dagger\mathbf{E}\mathbf{B}^\dagger - \mathbf{A}^\dagger\mathbf{C}\mathbf{M}^\dagger\mathbf{R}_A\mathbf{E}\mathbf{B}^\dagger - \mathbf{A}^\dagger\mathbf{S}\mathbf{C}^\dagger\mathbf{E}\mathbf{L}_B\mathbf{N}^\dagger\mathbf{D}\mathbf{B}^\dagger - \mathbf{A}^\dagger\mathbf{S}\mathbf{V}\mathbf{R}_N\mathbf{D}\mathbf{B}^\dagger + \mathbf{L}_A\mathbf{U} + \mathbf{Z}\mathbf{R}_B,$$

$$\mathbf{Y} = \mathbf{M}^\dagger\mathbf{R}_A\mathbf{E}\mathbf{D}^\dagger + \mathbf{L}_M\mathbf{S}^\dagger\mathbf{S}\mathbf{C}^\dagger\mathbf{E}\mathbf{L}_B\mathbf{N}^\dagger + \mathbf{L}_M(\mathbf{V} - \mathbf{S}^\dagger\mathbf{S}\mathbf{V}\mathbf{N}\mathbf{N}^\dagger) + \mathbf{W}\mathbf{R}_D,$$

where $\mathbf{U}$, $\mathbf{V}$, $\mathbf{Z}$ and $\mathbf{W}$ are arbitrary matrices of suitable sizes over $\mathbb{H}$, $\mathbf{M} := \mathbf{R}_A\mathbf{C}$, $\mathbf{N} := \mathbf{D}\mathbf{L}_B$, and $\mathbf{S} := \mathbf{C}\mathbf{L}_M$.

Using determinantal representations of the Moore-Penrose inverse, previously obtained in [**3**], within the framework of the theory of quaternion row-column determinants (introduced in [**4, 5**]), we got in [**6**] explicit determinantal representation formulas (analogs of Cramer's Rule) for the solution to Eq. (1) and to its special cases when its first term or both terms are one-sided. The Cramer's Rules for general, Hermitian, or η-Hermitian solutions ($\eta \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$) to the Sylvester-type matrix equations involving $*$-Hermicity or η-Hermicity (i.e. when in Eq. (1), $\mathbf{B} = \mathbf{A}^*$ and $\mathbf{D} = \mathbf{C}^*$, or $\mathbf{B} = \mathbf{A}^{\eta*}$ and $\mathbf{D} = \mathbf{C}^{\eta*}$, respectively) are derived in [**7**].

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