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Cramer's rules for Sylvester-type quaternion matrix equations

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Consider the two-sided generalized Sylvester matrix equation

$$\mathbf{AXB}+\mathbf{CYD}=\mathbf{E} \tag{1}$$

over the quaternion skew field $\mathbb{H}$. The Sylvester matrix equation has far reaching applications in different fields (see, e.g., [**1**]). Its solving is based on generalized inverses which are important tools in solving of matrix equations. Let for $\mathbf{A}\in\mathbb{H}^{m\times n}$, $\mathbf{A}^{\dagger}$ mean its Moore-Penrose generalized inverse, i.e. the exclusive matrix $\mathbf{X}\in\mathbb{H}^{n\times m}$ satisfying $\mathbf{AXA}=\mathbf{A}$, $\mathbf{XAX}=\mathbf{X}$, $(\mathbf{AX})^{*}=\mathbf{AX}$, $(\mathbf{XA})^{*}=\mathbf{XA}$. Furthermore, let $\mathbf{L}_A=\mathbf{I}-\mathbf{A}^{\dagger}\mathbf{A}$ and $\mathbf{R}_A=\mathbf{I}-\mathbf{A}\mathbf{A}^{\dagger}$ be a couple of projectors induced by $\mathbf{A}$. In [**2**] the solvability conditions to Eq. (1) was obtained and its general solution was expressed in terms of generalized inverses as follows:

$$\mathbf{X}=\mathbf{A}^{\dagger}\mathbf{E}\mathbf{B}^{\dagger}-\mathbf{A}^{\dagger}\mathbf{C}\mathbf{M}^{\dagger}\mathbf{R}_A\mathbf{E}\mathbf{B}^{\dagger}-\mathbf{A}^{\dagger}\mathbf{S}\mathbf{C}^{\dagger}\mathbf{E}\mathbf{L}_B\mathbf{N}^{\dagger}\mathbf{D}\mathbf{B}^{\dagger}-\mathbf{A}^{\dagger}\mathbf{S}\mathbf{V}\mathbf{R}_N\mathbf{D}\mathbf{B}^{\dagger}+\mathbf{L}_A\mathbf{U}+\mathbf{Z}\mathbf{R}_B,$$

$$\mathbf{Y}=\mathbf{M}^{\dagger}\mathbf{R}_A\mathbf{E}\mathbf{D}^{\dagger}+\mathbf{L}_M\mathbf{S}^{\dagger}\mathbf{S}\mathbf{C}^{\dagger}\mathbf{E}\mathbf{L}_B\mathbf{N}^{\dagger}+\mathbf{L}_M(\mathbf{V}-\mathbf{S}^{\dagger}\mathbf{S}\mathbf{V}\mathbf{N}\mathbf{N}^{\dagger})+\mathbf{W}\mathbf{R}_D,$$

where $\mathbf{U}$, $\mathbf{V}$, $\mathbf{Z}$ and $\mathbf{W}$ are arbitrary matrices of suitable sizes over $\mathbb{H}$, $\mathbf{M}:=\mathbf{R}_A\mathbf{C}$, $\mathbf{N}:=\mathbf{D}\mathbf{L}_B$, and $\mathbf{S}:=\mathbf{C}\mathbf{L}_M$.

Using determinantal representations of the Moore-Penrose inverse, previously obtained in [**3**], within the framework of the theory of quaternion row-column determinants (introduced in [**4, 5**]), we got in [**6**] explicit determinantal representation formulas (analogs of Cramer's Rule) for the solution to Eq. (1) and to its special cases when its first term or both terms are one-sided. The Cramer's Rules for general, Hermitian, or η-Hermitian solutions ($\eta\in\{\mathbf{i},\mathbf{j},\mathbf{k}\}$) to the Sylvester-type matrix equations involving $*$-Hermicity or η-Hermicity (i.e. when in Eq. (1), $\mathbf{B}=\mathbf{A}^{*}$ and $\mathbf{D}=\mathbf{C}^{*}$, or $\mathbf{B}=\mathbf{A}^{\eta *}$ and $\mathbf{D}=\mathbf{C}^{\eta *}$, respectively) are derived in [**7**].

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(z,k)-equivalence of matrices over Euclidean quadratic rings and solutions of matrix equation AX+YB=C

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Let $\mathbb{K} = \mathbb{Z}\left[\sqrt{k}\right]$ be a Euclidean quadratic ring, $e(a)$ be the Euclidean norm $a \in \mathbb{K}$ [**1**].

Definition 1. Matrices $A, B \in M(n, \mathbb{K})$ are called (z,k)-equivalent if there exist invertible matrices $S \in GL(n, \mathbb{Z})$ over the ring of integers $\mathbb{Z}$ and $Q \in GL(n, \mathbb{K})$ over quadratic ring $\mathbb{K}$ such that $A = SBQ$.

We established the standard form of matrices over a Euclidean quadratic ring with respect to the (z,k)-equivalence and used it to the description of the structure of solutions of the matrix equation $AX + YB = C$.

Theorem 1. *Let $D^A = diag\left(\mu_1^A, ..., \mu_n^A\right)$ be the Smith normal form of a matrix A. Then the matrix A is (z,k)-equivalent to the triangular form T^A with invariant factors μ_i^A, $i = 1, ..., n$ on the main diagonal that is*

$$SAQ = T^A = TD^A, \quad S \in GL(n, \mathbb{Z}), \quad Q \in GL(n, \mathbb{K}) \tag{1}$$

where $T = \left\|t_{ij}\right\|_1^n$ is the lower unitriangular matrix namely $t_{ij} = 0$ if $i < j$, $t_{ij} = 1$ if $i = j$ and $t_{ij} = 0$ if $\mu_i^A = 1$; $e(t_{ij}) < e(\mu_i^A)$ for $t_{ij} \neq 0$, $i, j = 1, ..., n$, $i > j$.

If $\mathbb{K}$ is a Euclidean imaginary quadratic ring, then the matrix A has a finite number of triangular form T^A in the form (1) with respect to (z,k)-equivalence.

Consider the matrix equation

$$AX + YB = C, \tag{2}$$

where $A, B, C \in M(n, \mathbb{K})$ are given matrices and $X, Y \in M(n, \mathbb{K})$ are unknown matrices. Let pair of matrices (A, B) be the (z,k)-equivalent to the pair $\left(T^A, T^B\right)$ of matrices T^A and T^B in the form (1) that is $SAQ_A = T^A$, $SBQ_B = T^B$, $S \in GL(n, \mathbb{Z})$, $Q_A, Q_B \in GL(n, \mathbb{K})$ [**2**]. Then from the equation (2) we get the equation

$$T^A H + W T^B = \tilde{C}, \tag{3}$$

where $H = Q_A^{-1} X Q_B$, $W = SYS^{-1}$, $\tilde{C} = SCQ_B$. The matrix equations (2) and (3) are equivalent. Thus the description of the solutions of equation (2) are reduced to the description of the solutions of equation (3).

Theorem 2. *If the equation* (3) *has a solution then it has such solutions $H = \left\|h_{ij}\right\|_1^n$, $W = \left\|w_{ij}\right\|_1^n$ that $h_{ij} = 0$ if $\mu_i = 1$, and $e(h_{ij}) < e(\mu_i^B)$ if $h_{ij} \neq 0$, $i, j = 1, ..., n$.*

If $\mathbb{K}$ is a Euclidean imaginary quadratic ring, then the equation (3) *has a finite number of such solutions.*