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(z,k)-equivalence of matrices over Euclidean quadratic rings and solutions of matrix equation AX+YB=C

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Let $\mathbb{K} = \mathbb{Z}\left[\sqrt{k}\right]$ be a Euclidean quadratic ring, $e(a)$ be the Euclidean norm $a \in \mathbb{K}$ [**1**].

Definition 1. Matrices $A, B \in M(n, \mathbb{K})$ are called (z,k)-equivalent if there exist invertible matrices $S \in GL(n, \mathbb{Z})$ over the ring of integers $\mathbb{Z}$ and $Q \in GL(n, \mathbb{K})$ over quadratic ring $\mathbb{K}$ such that $A = SBQ$.

We established the standard form of matrices over a Euclidean quadratic ring with respect to the (z,k)-equivalence and used it to the description of the structure of solutions of the matrix equation $AX + YB = C$.

Theorem 1. *Let $D^A = diag\left(\mu_1^A, ..., \mu_n^A\right)$ be the Smith normal form of a matrix A. Then the matrix A is (z,k)-equivalent to the triangular form T^A with invariant factors $\mu_i^A,\ i = 1, ..., n$ on the main diagonal that is*

$$SAQ = T^A = TD^A, \quad S \in GL(n, \mathbb{Z}), \quad Q \in GL(n, \mathbb{K}) \tag{1}$$

where $T = \left\|t_{ij}\right\|_1^n$ is the lower unitriangular matrix namely $t_{ij} = 0$ if $i < j,\ t_{ij} = 1$ if $i = j$ and $t_{ij} = 0$ if $\mu_i^A = 1;\ e(t_{ij}) < e(\mu_i^A)$ for $t_{ij} \neq 0,\ i, j = 1, ..., n,\ i > j$.

If $\mathbb{K}$ is a Euclidean imaginary quadratic ring, then the matrix A has a finite number of triangular form T^A in the form (1) with respect to (z,k)-equivalence.

Consider the matrix equation

$$AX + YB = C, \tag{2}$$

where $A, B, C \in M(n, \mathbb{K})$ are given matrices and $X, Y \in M(n, \mathbb{K})$ are unknown matrices. Let pair of matrices (A, B) be the (z,k)-equivalent to the pair $\left(T^A, T^B\right)$ of matrices T^A and T^B in the form (1) that is $SAQ_A = T^A$, $SBQ_B = T^B$, $S \in GL(n, \mathbb{Z})$, $Q_A, Q_B \in GL(n, \mathbb{K})$ [**2**]. Then from the equation (2) we get the equation

$$T^A H + W T^B = \tilde{C}, \tag{3}$$

where $H = Q_A^{-1} X Q_B$, $W = SYS^{-1}$, $\tilde{C} = SCQ_B$. The matrix equations (2) and (3) are equivalent. Thus the description of the solutions of equation (2) are reduced to the description of the solutions of equation (3).

Theorem 2. *If the equation* (3) *has a solution then it has such solutions $H = \left\|h_{ij}\right\|_1^n$, $W = \left\|w_{ij}\right\|_1^n$ that $h_{ij} = 0$ if $\mu_i = 1$, and $e(h_{ij}) < e(\mu_i^B)$ if $h_{ij} \neq 0,\ i, j = 1, ..., n$.*

If $\mathbb{K}$ is a Euclidean imaginary quadratic ring, then the equation (3) *has a finite number of such solutions.*

References

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Separability of the lattice of τ-closed totally ω-composition formations of finite groups

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All groups considered are finite. All notations and terminologies are standard [**1**]-[**5**].

Let ω be a non-empty set of primes. Every function of the form $f : \omega \bigcup \{\omega'\} \to \{\text{formations}\}$ is called an *ω-composition satellite.* For any ω-composition satellite $CF_\omega(f) = \{G | G/R_\omega(G) \in f(\omega')$ and $G/C^p(G) \in f(p)$ for all $p \in \pi(Com(G)) \cap \omega\}$. If the formation $\mathfrak{F}$ is such that $\mathfrak{F} = CF_\omega(f)$ for some ω-composition satellite f, then it is ω-composition formation, and f — ω-composition satellite of this formation.

Every formation of groups is called *0-multiply ω-composition.* For $n \geq 1$, a formation $\mathfrak{F}$ is called *n-multiply ω-composition*, if it has an ω-composition satellite f such that every value $f(p)$ of f is an $(n-1)$-multiply ω-composition formation. A formation $\mathfrak{F}$ is called *totally ω-composition* if it is n-multiply ω-composition for all natural n.

Let for any group G, $\tau(G)$ be a set of subgroups of G such that $G \in \tau(G)$. Then we say following [**5**] that τ is a *subgroup functor* if for every epimorphism $\varphi : A \to B$ and any groups $H \in \tau(A)$ and $T \in \tau(B)$ we have $H^\varphi \in \tau(B)$ and $T^{\varphi^{-1}} \in \tau(A)$. A class $\mathfrak{F}$ of groups is called *τ-closed* if $\tau(G) \subseteq \mathfrak{F}$ for all $G \in \mathfrak{F}$.

Let $\mathfrak{X}$ be some set of groups. Then $c^\tau_{\omega_\infty}$form $\mathfrak{X}$ denotes the totally ω-composition formation generated by $\mathfrak{X}$, i.e. $c^\tau_{\omega_\infty}$form$\mathfrak{X}$ is the intersection of all τ-closed totally ω-composition formations containing $\mathfrak{X}$. For any two τ-closed totally ω-composition formations $\mathfrak{M}$ and $\mathfrak{H}$, we write $\mathfrak{M} \vee^\tau_{\omega_\infty} \mathfrak{H} = c^\tau_{\omega_\infty}\text{form}(\mathfrak{M} \cup \mathfrak{H})$.

With respect to the operations $\vee^\tau_{\omega_\infty}$ and $\cap$ the set $c^\tau_{\omega_\infty}$ of all τ-closed totally ω-composition formations forms a complete lattice. Formations in $c^\tau_{\omega_\infty}$ are called *$c^\tau_{\omega_\infty}$-formations.*

Let $\mathfrak{X}$ be a non-empty class of finite groups. A complete lattice θ of formations is called $\mathfrak{X}$-separable, if for every term $\nu(x_1, ..., x_n)$ of signature $\{\cap, \vee_\theta\}$, θ-formations $\mathfrak{F}_1$, ..., $\mathfrak{F}_n$ and every group $A \in \mathfrak{X} \cap \nu(\mathfrak{F}_1, ..., \mathfrak{F}_n)$ are exists $\mathfrak{X}$-groups $A_1 \in \mathfrak{F}_1, ..., A_n \in \mathfrak{F}_n$ such that $A \in \nu(\theta\text{form}A_1, ..., \theta\text{form}A_n)$. In particular, if $\mathfrak{X} = \mathfrak{G}$ is the class of all finite groups then the lattice θ of formations is called $\mathfrak{G}$-separable or separable.

Theorem 1. *The lattice $c^\tau_{\omega_\infty}$ all τ-closed totally ω-composition formations is $\mathfrak{G}$-separated.*

Let τ be the trivial subgroup functor or let ω be the set of all primes. Then we obtain