Residual and fixed modules

Vasyl Petechuk, Yulia Petechuk

Let V be an arbitrary $R-$module over an associative ring R of 1, $GL(V)$ is a group of automorphisms of module V.

The $R(\sigma) = (\sigma - 1)V$ and $P(\sigma) = ker(\sigma - 1)$ respectively, are called residual and fixed submodules of the module V of the endomorphism σ.

Inclusions system

$$\begin{cases} R(\sigma_1) \in P(\sigma_2); \\ R(\sigma_2) \in P(\sigma_1) \end{cases} \quad (1)$$

exists if and only if $(\sigma_1 - 1)(\sigma_2 - 1) = (\sigma_2 - 1)(\sigma_1 - 1) = 0$ otherwise when $\sigma_1\sigma_2 = \sigma_2\sigma_1 = \sigma_1 + \sigma_2 - 1$.

It is clear that the commutativity $\sigma_1\sigma_2 = \sigma_2\sigma_1$ follows from the system (1). On the contrary, it is not always true. It is easy to see that if $\sigma_1\sigma_2 = \sigma_2\sigma_1$ and one of the inclusions of the system (1) takes place, then the second inclusion of system (1) also takes place. If $\sigma_1\sigma_2 = \sigma_2\sigma_1$ and $R(\sigma_1) \bigcap R(\sigma_2) = 0$ or $P(\sigma_1) + P(\sigma_2) = V$ then system (1) takes place. Finding other conditions for which of the commutativity σ_1 and σ_2 follows system (1) is the main purpose of the work.

Properties of residual and fixed submodules are used to describe homomorphisms of matrix groups over associative rings from 1 [**1**]. The method of residual and fixed subspaces was introduced by O'Meara. A shorter version of the proof of O'Meara-Sosnovskij theorem, which describes isomorphisms between full groups preserves projective transvections, has proposed by one of the authors in [**3**].

The basis of the method of residual and fixed subspaces is the two main properties of transvection. In particular, if σ_1 and σ_2 are transvections, then $\sigma_1\sigma_2 = \sigma_2\sigma_1$ if and only if there is a system (1), and in the case where $R(\sigma_1) \subseteq R(\sigma_2)$ and $R(\sigma_2) \subseteq R(\sigma_1)$, then the commutator $[\sigma_1, \sigma_2]$ is a transvection with a residual subspace $R(\sigma_1)$ and a fixed subspace $P(\sigma_2)$.

In [**2**] it is proved that if R is a division ring, V is a finite-dimensional vector space over R, $dimR(\sigma_1) = dimR(\sigma_2) = 2$, $R(\sigma_1) \bigcap P(\sigma_1) = 0$, σ_2 is a unipotent element of level 2 or $dimR(\sigma_1) = 2$, $R(\sigma_1) \bigcap P(\sigma_1) \neq 0$, σ_2 is a transvection then $\sigma_1\sigma_2 = \sigma_2\sigma_1$ if and only if there is (1).

Authors are proven

Theorem. *Let R be a division ring, V is a finite-dimensional vector space over R, $dimR(\sigma_1) \bigcap P(\sigma_1) \neq dimR(\sigma_1) - 1$, σ_2 is a transvection. Equation $\sigma_1\sigma_2 = \sigma_2\sigma_1$ is executed if and only if system* (1) *takes place.*

The condition of the theorem on σ_1 means that $R(\sigma_1) \subseteq P(\sigma_1)$ or $R(\sigma_1) \bigcap P(\sigma_1)$ is a hyperplane in $R(\sigma_1)$.

We emphasize that if $dimR(\sigma_1) < 2$, then the conditions of the theorem are fulfilled automatically. If $dimR(\sigma_1) \geq 2$, then without the assumption $dimR(\sigma_1) \bigcap P(\sigma_1) \geq dimR(\sigma_1) - 1$ the theorem does not hold.

This shows an example $\sigma_1 = diag(\alpha, \ldots, \alpha, 1, \ldots, 1)$, $\sigma_2 = t_1k(1)$, where α is taken k times, $\alpha \neq 0$, $\alpha \neq 1$, $k \geq 2$.

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Contact information

Vasyl Petechuk
Department of Mathematics and Informatics, Institute of Postgraduate Education, City Uzhgorod, Ukraine
Email address: vasil.petechuk@gmail.com

Yulia Petechuk
Department of Mathematics and Informatics, Transcarpathian Hungarian Institute by Ferenc Rakoczy II, City Beregovo, Ukraine
Email address: vasil.petechuk@gmail.com

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Method of residual and fixed subspaces was introduced by O'Meara.

Solvable Lie algebras of derivations of rank one

Anatoliy Petravchuk, Kateryna Sysak

Let $\mathbb{K}$ be a field of characteristic zero and $A = \mathbb{K}[x_1, \ldots, x_n]$ the polynomial ring over $\mathbb{K}$. A $\mathbb{K}$-derivation D of A is a $\mathbb{K}$-linear mapping $D\colon A \to A$ that satisfies the rule: $D(ab) = D(a)b+aD(b)$ for all $a,\ b \in A$. The set $W_n(\mathbb{K})$ of all $\mathbb{K}$-derivations of the polynomial ring A forms a Lie algebra over $\mathbb{K}$. This Lie algebra is simultaneously a free module over A with the standard basis $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\}$. Therefore, for each subalgebra L of $W_n(\mathbb{K})$ one can define the rank $\operatorname{rank}_A L$ of L over the ring A. Note that for any $f \in A$ and $D \in W_n(\mathbb{K})$ a derivation fD is defined by the rule: $fD(a) = f \cdot D(a)$ for all $a \in A$.

Finite dimensional subalgebras L of $W_n(\mathbb{K})$ such that $\operatorname{rank}_A L = 1$ were described in [**1**]. We study solvable subalgebras $L \subseteq W_n(\mathbb{K})$ of rank 1 over A without restrictions on the dimension over the field $\mathbb{K}$.

Recall that a polynomial $f \in A$ is said to be a Darboux polynomial for a derivation $D \in W_n(\mathbb{K})$ if $f \neq 0$ and $D(f) = \lambda f$ for some polynomial $\lambda \in A$. The polynomial λ is called the polynomial eigenvalue of f for the derivation D. Some properties of Darboux polynomials and their applications in the theory of differential equations can be found in [**3**]. Denote by A_D^λ the set of all Darboux polynomials for $D \in W_n(\mathbb{K})$ with the same polynomial eigenvalue λ and of the zero polynomial. Obviously, the set A_D^λ is a vector space over $\mathbb{K}$. If V is a subspace of A_D^λ for any derivation $D \in W_n(\mathbb{K})$, then we denote by VD the set of all derivations fD, $f \in V$.

Theorem 1. *Let L be a subalgebra of the Lie algebra $W_n(\mathbb{K})$ of rank 1 over A and $\dim_{\mathbb{K}} L \geq 2$. The Lie algebra L is abelian if and only if there exist a derivation $D \in W_n(\mathbb{K})$ and a Darboux polynomial f for D with the polynomial eigenvalue λ such that $L = VD$ for some $\mathbb{K}$-subspace $V \subseteq A_D^\lambda$.*

Using this result one can characterize nonabelian subalgebras of rank 1 over A of the Lie algebra $W_n(\mathbb{K})$. For the Lie algebra $\widetilde{W}_n(\mathbb{K})$ of all $\mathbb{K}$-derivations of the field $\mathbb{K}(x_1, x_2, \ldots, x_n)$ this problem is simpler and was considered in [**2**].

References

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