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Method of residual and fixed subspaces was introduced by O'Meara.

Solvable Lie algebras of derivations of rank one

Anatoliy Petravchuk, Kateryna Sysak

Let $\mathbb{K}$ be a field of characteristic zero and $A = \mathbb{K}[x_1, \dots, x_n]$ the polynomial ring over $\mathbb{K}$. A $\mathbb{K}$-derivation D of A is a $\mathbb{K}$-linear mapping $D\colon A \to A$ that satisfies the rule: $D(ab) = D(a)b + aD(b)$ for all $a,\ b \in A$. The set $W_n(\mathbb{K})$ of all $\mathbb{K}$-derivations of the polynomial ring A forms a Lie algebra over $\mathbb{K}$. This Lie algebra is simultaneously a free module over A with the standard basis $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\}$. Therefore, for each subalgebra L of $W_n(\mathbb{K})$ one can define the rank $\operatorname{rank}_A L$ of L over the ring A. Note that for any $f \in A$ and $D \in W_n(\mathbb{K})$ a derivation fD is defined by the rule: $fD(a) = f \cdot D(a)$ for all $a \in A$.

Finite dimensional subalgebras L of $W_n(\mathbb{K})$ such that $\operatorname{rank}_A L = 1$ were described in [**1**]. We study solvable subalgebras $L \subseteq W_n(\mathbb{K})$ of rank 1 over A without restrictions on the dimension over the field $\mathbb{K}$.

Recall that a polynomial $f \in A$ is said to be a Darboux polynomial for a derivation $D \in W_n(\mathbb{K})$ if $f \neq 0$ and $D(f) = \lambda f$ for some polynomial $\lambda \in A$. The polynomial λ is called the polynomial eigenvalue of f for the derivation D. Some properties of Darboux polynomials and their applications in the theory of differential equations can be found in [**3**]. Denote by A_D^λ the set of all Darboux polynomials for $D \in W_n(\mathbb{K})$ with the same polynomial eigenvalue λ and of the zero polynomial. Obviously, the set A_D^λ is a vector space over $\mathbb{K}$. If V is a subspace of A_D^λ for any derivation $D \in W_n(\mathbb{K})$, then we denote by VD the set of all derivations fD, $f \in V$.

Theorem 1. *Let L be a subalgebra of the Lie algebra $W_n(\mathbb{K})$ of rank 1 over A and* $\dim_{\mathbb{K}} L \geq 2$. *The Lie algebra L is abelian if and only if there exist a derivation $D \in W_n(\mathbb{K})$ and a Darboux polynomial f for D with the polynomial eigenvalue λ such that $L = VD$ for some $\mathbb{K}$-subspace $V \subseteq A_D^\lambda$.*

Using this result one can characterize nonabelian subalgebras of rank 1 over A of the Lie algebra $W_n(\mathbb{K})$. For the Lie algebra $\widetilde{W}_n(\mathbb{K})$ of all $\mathbb{K}$-derivations of the field $\mathbb{K}(x_1, x_2, \dots, x_n)$ this problem is simpler and was considered in [**2**].

References

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Classification of quasigroups according to their parastrophic symmetry groups

Yevhen Pirus

Let Q be a set, a mapping $f : Q^3 \to Q$ is called an invertible ternary operation (=function), if it is invertible element in all semigroups $(\mathcal{O}_3; \underset{0}{\oplus})$, $(\mathcal{O}_3; \underset{1}{\oplus})$ and $(\mathcal{O}_3; \underset{2}{\oplus})$, where $\mathcal{O}_3$ is the set of all ternary operations defined on Q and

$$(f \underset{1}{\oplus} f_1)(x_1, x_2, x_3) := f(f_1(x_1, x_2, x_3), x_2, x_3), \quad (f \underset{2}{\oplus} f_1)(x_1, x_2, x_3) := f(x_1, f_1(x_1, x_2, x_3), x_3),$$

$$(f \underset{3}{\oplus} f_1)(x_1, x_2, x_3) := f(x_1, x_2, f_1(x_1, x_2, x_3)).$$

The set of all ternary invertible functions is denoted by Δ_3. If an operation f is invertible and ${}^{(14)}f$, ${}^{(24)}f$, ${}^{(34)}f$ are its inverses in those semigroups, then the algebra $(Q; f, {}^{(14)}f, {}^{(24)}f, {}^{(34)}f)$ (in brief, $(Q; f)$) is called a *ternary quasigroup* [**1**]. The inverses are also invertible. All inverses to inverses are called *σ-parastrophes* of the operation f and can be defined by

$${}^{\sigma}f(x_{1\sigma}, x_{2\sigma}, x_{3\sigma}) = x_{4\sigma} \quad :\Leftrightarrow \quad f(x_1, x_2, x_3) = x_4, \quad \sigma \in S_4,$$

where S_4 denotes the group of all bijections of the set $\{0, 1, 2, 3\}$. Therefore in general, every invertible operation has 24 parastrophes. Since parastrophes of a quasigroup satisfy the equalities ${}^{\sigma}({}^{\tau}f) = {}^{\sigma\tau}f$, then the symmetric group S_4 defines an action on the set Δ_3. In particular, the fact implies that the number of different parastrophes of an invertible operation is a factor of 24. More precisely, it is equal to $24/|\mathrm{Ps}(f)|$, where $\mathrm{Ps}(f)$ denotes a stabilizer group of f under this action which is called *parastrophic symmetry group* of the operation f.

Let $\mathfrak{P}(H)$ denote the class of all quasigroups whose parastrophic symmetry group contains the group $H \in S_4$. A ternary quasigroup $(Q; f)$ belongs to $\mathfrak{P}(H)$ if and only if ${}^{\sigma}f = f$ for all σ from a set G of generators of the group H, therefore, the class of quasigroup $\mathfrak{P}(H)$ is a variety.

For every subgroup H of the group S_4 the variety $\mathfrak{P}(H)$ are described and its subvariety of ternary group isotopes are found. For example, let

$$D_8 := \{\iota, (12), (34), (12)(34), (13)(24), (14)(23), (1324), (1423)\} \leq S_4.$$

Theorem 1. *A ternary quasigroup $(Q; f)$ belong to the variety $\mathfrak{P}(D_8)$ if and only if*

$$f(x, y, z) = f(y, x, z), \quad f(x, y, f(x, y, z)) = z, \quad f(z, f(x, y, z), x) = y. \tag{1}$$