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On the conjugate sets of IP-quasigroups

Tatiana Rotari

A quasigroup (Q, A) is called quasigroup with the property of invertibility *(an IP-quasigroup)* if there exist two mappings I_l an I_r on the set Q into Q that $A(I_l x,\, A(x,\, y)) = y$ and $A(A(y,\, x),\, I_r x) = y$ for any $x, y \in Q$ [1]. The mappings I_l and I_r are permutations and $I_l^2 = I_r^2 = \varepsilon$.

It is known that the system Σ of six (not necessarily distinct) conjugates (or parastrophes): A, ${}^{r}A$, ${}^{l}A$, ${}^{rl}A$, ${}^{lr}A$, ${}^{s}A$, where ${}^{r}A(x,y) = z \Leftrightarrow A(x,z) = y$, ${}^{l}A(x,y) = z \Leftrightarrow A(z,y) = x$, ${}^{s}A(x,y) = A(y,x)$ (${}^{rl}A = {}^{r}({}^{l}A)$) corresponds to a quasigroup (Q, A).

It is known [**2**] that the number of distinct conjugates in Σ can be 1, 2, 3 or 6.

Using suitable Belousov's designation of conjugates of a quasigroup (Q, A) from [**1**] we have the following system Σ of conjugates:

$$\Sigma = \left\{A,\ {}^{r}A,\ {}^{l}A,\ {}^{lr}A,\ {}^{rl}A,\ {}^{s}A\right\},$$

where ${}^{1}A = A$, ${}^{r}A = A^{-1}$, ${}^{l}A = {}^{-1}A$, ${}^{lr}A = {}^{-1}(A^{-1})$, ${}^{rl}A = ({}^{-1}A)^{-1}$, ${}^{s}A = A^{*}$.

Note that

$$\left({}^{-1}(A^{-1})\right)^{-1} = {}^{rlr}A = {}^{-1}\left(({}^{-1}A)^{-1}\right) = {}^{lrl}A = {}^{s}A$$

and ${}^{rr}A = {}^{ll}A = A$, ${}^{\sigma\tau}A = {}^{\sigma}({}^{\tau}A)$.

The conjugates og IP-quasigroup have the following form [1, 4]:

$${}^{l}A(x,\, y) = A(x,\, I_r y),\ {}^{r}A(x,\, y) = A(I_l x,\, y),\ {}^{lr}A(x,\, y) = I_l A(x,\, I_l y),$$

$${}^{rl}A(x,\, y) = I_r A(I_l x,\, y),\ {}^{s}A(x,\, y) = I_l A(I_r x,\, I_r y).$$

The following Theorem 1 of [**3, 4**] describes all possible conjugate sets for quasigroups and points out the only possible variants of equality of conjugates:

Theorem 1. *The following conjugate sets of a quasigroups $(Q,\, A)$ are only possible:* $\overline{\Sigma}_1(A) = \{A\}$, $\overline{\Sigma}_2 = \{A,\, {}^{s}A\} = \{A = {}^{lr}A = {}^{rl}A,\, {}^{l}A = {}^{r}A = {}^{s}A\}$, $\overline{\Sigma}_6 = \{A,\, {}^{l}A,\, {}^{r}A,\, {}^{lr}A,\, {}^{rl}A,\, {}^{s}A\}$, $\overline{\Sigma}_3 = \{A,\, {}^{lr}A,\, {}^{rl}A\}$ *and three cases are only possible:* $\overline{\Sigma}_3^1 = \{A = {}^{r}A,\, {}^{l}A = {}^{lr}A,\, {}^{rl}A = {}^{s}A\}$; $\overline{\Sigma}_3^2 = \{A = {}^{l}A,\, {}^{r}A = {}^{rl}A,\, {}^{lr} = {}^{s}A\}$; $\overline{\Sigma}_3^3 = \{A = {}^{s}A,\, {}^{r}A = {}^{lr}A,\, {}^{l}A = {}^{rl}A\}$.

We study the conjugate sets on the distict conjugates of IP-quasigroups and IP-loops.

Theorem 2. *Let a quasigroup $(Q,\, A)$ be an IP-quasigroup. Then*
$\Sigma(A) = \overline{\Sigma}_1(A)$ *if and only if* $I_r = I_l = I = \varepsilon$;
$\Sigma(A) = \overline{\Sigma}_2(A)$ *if and only if* $I_l = I_r = I \neq \varepsilon$, $A(x,\, y) \neq A(y,\, x)$ *and* $IA(x,\, y) = A(y,\, x)$;
$\Sigma(A) = \overline{\Sigma}_3^1(A)$ *if and only if* $I_l = \varepsilon \neq I_r$;
$\Sigma(A) = \overline{\Sigma}_3^2(A)$ *if and only if* $I_r = \varepsilon \neq I_l$;
$\Sigma(A) = \overline{\Sigma}_3^3(A)$ *if and only if* $I_l = I_r = I \neq \varepsilon$ *and* $A(x,\, y) = A(y,\, x)$;
$\Sigma(A) = \overline{\Sigma}_6(A)$ *if and only if* $I_l = I_r = I \neq \varepsilon$, $A(x,\, y) \neq A(y,\, x)$ *and* $IA(x,\, y) \neq A(y,\, x)$.;

A special case of IP-loops is a Moufang loop defined by the identity

$$A(x,\ A(y,\ A(x,\ z))) = A(A(A(x,\ y), x), z).$$

From the Theoreme the following corollary easy follow.

Corollary 1. *Let $(Q,\ \ A)$ be an IP-loop (a Moufang loop), then*
$\Sigma(A) = \overline{\Sigma}_1(A)$ *if and only if* $I = \varepsilon$;
$\Sigma(A) = \overline{\Sigma}_3^3(A)$ *if and only if* $(Q,\ A)$ *is commutative and* $I \neq \varepsilon$;
$\Sigma(A) = \overline{\Sigma}_6(A)$ *if and only if* $(Q,\ A)$ *is noncommutative.*

Note that the case $\Sigma(A) = \overline{\Sigma}_2(A)$ ($\overline{\Sigma}_3^1(A)$ or $\overline{\Sigma}_3^2(A)$) for any IP-loops is impossible.

References

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On sublattices of the lattice of multiply saturated formations of finite groups

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All groups considered are finite. We use terminology and notations from [**1**]–[**3**].

Let σ be some partition of the set of all primes $\mathbb{P}$, that is, $\sigma = \{\sigma_i | i \in I\}$, where $\mathbb{P} = \bigcup_{i\in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$; $\Pi \subseteq \sigma$ and $\Pi' = \sigma \setminus \Pi$. If n is an integer, the symbol $\sigma(n)$ denotes the set $\{\sigma_i | \sigma_i \cap \pi(n) \neq \emptyset\}$, if G is a finite group, then $\sigma(G) = \sigma(|G|)$, and if $\mathfrak{F}$ is a class of groups, then $\sigma(\mathfrak{F}) = \bigcup_{G\in\mathfrak{F}} \sigma(G)$.

A function f of the form $f : \sigma \to \{\text{formations of groups}\}$ is called *a formation σ-function.* For any formation σ-function f the symbol $LF_\sigma(f)$ denotes the class

$$LF_\sigma(f) = (G \mid G = 1 \text{ or } \ G \neq 1 \text{ and } \ G/O_{\sigma_i',\sigma_i}(G) \in f(\sigma_i) \text{ for all } \sigma_i \in \sigma(G)).$$

If for some formation σ-function f we have $\mathfrak{F} = LF_\sigma(f)$, then we say, that the class $\mathfrak{F}$ is *σ-local* and f is a *σ-local definition of* $\mathfrak{F}$.

We suppose that every formation of groups is *0-multiply σ-local*; for $n \geq 1$, we say that the formation $\mathfrak{F}$ is *n-multiply σ-local* provided either $\mathfrak{F} = (1)$ is the formation of all identity groups or $\mathfrak{F} = LF_\sigma(f)$, where $f(\sigma_i)$ is $(n-1)$-multiply σ-local for all $\sigma_i \in \sigma(\mathfrak{F})$. The formation $\mathfrak{F}$ is said to be *totally σ-local* provided $\mathfrak{F}$ it is n-multiply σ-local for all $n \in \mathbb{N}$.

In the classical case, when $\sigma = \sigma^1 = \{\{2\}, \{3\}, \ldots\}$, a formation σ-function, a σ-local formation and an n-multiply σ-local formation are, respectively, a formation function, a local formation (a saturated formation), and an n-multiply local formation (an n-multiply saturated formation) in the usual sense [**4**]–[**6**].