Let (F_1, F_2, F_3) be the lexicographical sequence of functional variables of a ternary generalized functional equation $T_1 = T_2$ of length three. A sequence (f_1, f_2, f_3) of invertible ternary functions defined on a carrier is called a *solution* of $T_1 = T_2$ if substituting f_1 for F_1, f_2 for F_2 and f_3 for F_3, we obtain a true proposition $t_1 = t_2$, i.e., $t_1 = t_2$ is an identity [**2**].

The classification theorem of generalized ternary quadratic quasigroup functional equations of length three is given in [**3**]. There are four non-equivalent functional equations. The general solution of one of them is given in the following theorem. Solutions of the other three equations are formulated in [**4**].

Theorem 1. *A triplet (f_1, f_2, f_3) of ternary invertible operations defined on a set Q is a solution of the functional equation*

$$F_1(F_2(x, y, z), x, u) = F_3(y, z, u)$$

if and only if there exist binary invertible operations $\circ$, $*$, $\diamond$ on Q such that

$$f_1(y, x, u) = (x \diamond y) * u, \qquad f_2(x, y, z) = x \overset{r}{\diamond} (y \circ z), \qquad f_3(y, z, u) = (y \circ z) * u.$$

References

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Some conditions for a quasigroup to be a group isotope

Fedir Sokhatsky

A binary quasigroup is a pair $(Q; \circ)$, where Q is a set called a *carrier* and $\circ$ is an invertible binary operation defined on Q, i.e., there exist operations $\overset{\ell}{\circ}$ and $\overset{r}{\circ}$ such that for any x, $y \in Q$

$$(x \overset{\ell}{\circ} y) \circ y = x, \qquad (x \circ y) \overset{\ell}{\circ} y = x, \qquad x \circ (x \overset{r}{\circ} y) = y, \qquad x \overset{r}{\circ} (x \circ y) = y.$$

We say that an identity has a *group isotope property*, if every quasigroup satisfying this identity is isotopic to a group.

Definition 1. We say that variables $x_1, \ldots, x_n$ are *isolated in an identity* $\omega = v$ by sub-terms $t_1, \ldots, t_k$, if all appearances in the identity of the variables belong to two of these terms and every variable has one appearance in at least one of the terms.

Let x, y, z be arbitrary fixed variables. We will write $t(x, y)$ if the term t contains the variables x and y and does not contain z.

Theorem 1. *A quasigroup identity has a group isotopic property if three of its variables x, y, z are isolated by some sub-terms $t_1(x, y)$, $t_2(x, z)$, $t_3(y, z)$.*

For example, each quasigroup satisfying the identity

$$\Big((u^{n_1}((\underbrace{x\cdot(xu)^{n_2}y)\cdot x^{n_3}}_{t_1(x,y)})\cdot u^{n_4})\cdot(v\cdot(\underbrace{z^{n_5}x\cdot zu}_{t_3(x,z)})v)u)\Big)\cdot(\underbrace{y\cdot(zn^{n_6})y^n}_{t_2(y,z)}))=v$$

is isotopic to a group. A bracketing in u^{n_1}, $(xu)^{n_2}$,... does not matter.

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Canonical decompositions of solutions of functional equation of generalized mediality

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Let Q be a set, a mapping $f : Q^2 \to Q$ is called an invertible binary operation (=function), if it is invertible element in both semigroups $(\mathcal{O}_2; \underset{0}{\oplus})$ and $(\mathcal{O}_2; \underset{1}{\oplus})$, where $\mathcal{O}_2$ is the set of all binary operations defined on Q and

$$(f \underset{0}{\oplus} g)(x, y) := f(g(x, y), y), \qquad (f \underset{1}{\oplus} g)(x, y) := f(x, g(x, y)).$$

The set of all binary invertible functions is denoted by Δ_2. A functional equation

$$F_1(F_2(x, y), F_3(u, v)) = F_4(F_5(x, u), F_6(y, v)), \tag{1}$$

where $F_1, \ldots, F_6$ are functional variables and x, y, u, v are individual variables, is called a *functional equation of generalized mediality*. The equation was solved in [**1**]. Namely, the following theorem was proved

Theorem 1. *A sequence* $(f_1, \ldots, f_6)$ *of invertible functions defined on a set* Q *is a solution of* (1) *if and only if there exists a comutative group* $(Q; +, 0)$ *and bijections* $\alpha_1, \ldots, \alpha_6$ *of* Q *such that*

$$f_1(x, z) = \alpha_5 x + \alpha_6 z, \qquad f_2(x, y) = {\alpha_5}^{-1}(\alpha_1 x + \alpha_2 y), \qquad f_3(u, v) = {\alpha_6}^{-1}(\alpha_3 u + \alpha_4 v),$$

$$f_4(z, y) = \alpha_7 z + \alpha_8 y, \qquad f_5(x, u) = {\alpha_7}^{-1}(\alpha_1 x + \alpha_3 u), \qquad f_6(y, v) = {\alpha_8}^{-1}(\alpha_2 y + \alpha_4 v).$$

The sequence $(+, \alpha_1, \ldots, \alpha_8)$ will be called a *decomposition* of the solution $(f_1, \ldots, f_6)$. Theorem 1 proves that every solution has a decomposition and moreover every sequence uniquely defines a solution of (1). But the same solution may have different decomposition. For example, let θ be an arbitrary automorphism of the group $(Q; +)$, it is easy to see that the sequence $(+, \theta\alpha_1, \ldots, \theta\alpha_8)$ defines the same solution of (1).

A decomposition $(+, \alpha_1, \ldots, \alpha_8)$ of a solution of (1) will be called *0-canonical* if 0 is a neutral element of the group $(Q; +)$ and $\alpha_1 0 = \alpha_5 0 = \alpha_7 0 = 0$.

Theorem 2. *Every element* $0 \in Q$ *uniquely defines a canonical decomposition of an arbitrary solution of the functional equation of generalized mediality.*