For example, each quasigroup satisfying the identity

$$\Big((u^{n_1}((\underbrace{x\cdot(xu)^{n_2}y)\cdot x^{n_3}}_{t_1(x,y)})\cdot u^{n_4})\cdot(v\cdot(\underbrace{z^{n_5}x\cdot zu}_{t_3(x,z)})v)u)\Big)\cdot(\underbrace{y\cdot(zn^{n_6})y^{n}}_{t_2(y,z)}))=v$$

is isotopic to a group. A bracketing in u^{n_1}, $(xu)^{n_2}$,... does not matter.

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Canonical decompositions of solutions of functional equation of generalized mediality

Fedir Sokhatsky, Diana Kirka

Let Q be a set, a mapping $f : Q^2 \to Q$ is called an invertible binary operation (=function), if it is invertible element in both semigroups $(\mathcal{O}_2; \underset{0}{\oplus})$ and $(\mathcal{O}_2; \underset{1}{\oplus})$, where $\mathcal{O}_2$ is the set of all binary operations defined on Q and

$$(f \underset{0}{\oplus} g)(x, y) := f(g(x, y), y), \qquad (f \underset{1}{\oplus} g)(x, y) := f(x, g(x, y)).$$

The set of all binary invertible functions is denoted by Δ_2. A functional equation

$$F_1(F_2(x, y), F_3(u, v)) = F_4(F_5(x, u), F_6(y, v)), \tag{1}$$

where $F_1, \ldots, F_6$ are functional variables and x, y, u, v are individual variables, is called a *functional equation of generalized mediality.* The equation was solved in [**1**]. Namely, the following theorem was proved

Theorem 1. *A sequence $(f_1, \ldots, f_6)$ of invertible functions defined on a set Q is a solution of* (1) *if and only if there exists a comutative group $(Q; +, 0)$ and bijections $\alpha_1, \ldots, \alpha_6$ of Q such that*

$$f_1(x, z) = \alpha_5x + \alpha_6z, \qquad f_2(x, y) = {\alpha_5}^{-1}(\alpha_1x + \alpha_2y), \qquad f_3(u, v) = {\alpha_6}^{-1}(\alpha_3u + \alpha_4v),$$

$$f_4(z, y) = \alpha_7z + \alpha_8y, \qquad f_5(x, u) = {\alpha_7}^{-1}(\alpha_1x + \alpha_3u), \qquad f_6(y, v) = {\alpha_8}^{-1}(\alpha_2y + \alpha_4v).$$

The sequence $(+, \alpha_1, \ldots, \alpha_8)$ will be called a *decomposition* of the solution $(f_1, \ldots, f_6)$. Theorem 1 proves that every solution has a decomposition and moreover every sequence uniquely defines a solution of (1). But the same solution may have different decomposition. For example, let θ be an arbitrary automorphism of the group $(Q; +)$, it is easy to see that the sequence $(+, \theta\alpha_1, \ldots, \theta\alpha_8)$ defines the same solution of (1).

A decomposition $(+, \alpha_1, \ldots, \alpha_8)$ of a solution of (1) will be called *0-canonical* if 0 is a neutral element of the group $(Q; +)$ and $\alpha_1 0 = \alpha_5 0 = \alpha_7 0 = 0$.

Theorem 2. *Every element $0 \in Q$ uniquely defines a canonical decomposition of an arbitrary solution of the functional equation of generalized mediality.*

Canonical decompositions of solutions of the functional equations of generalized associativity are found in [**2**].

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About group isotopes with inverse property

Fedir Sokhatsky, Alla Lutsenko

A *quasigroup* is an algebra $(Q;\cdot;\overset{\ell}{\cdot};\overset{r}{\cdot})$ with identities

$$(x\cdot y)\overset{\ell}{\cdot} y=x,\qquad (x\overset{\ell}{\cdot} y)\cdot y=x,\qquad x\overset{r}{\cdot}(x\cdot y)=y,\qquad x\cdot(x\overset{r}{\cdot} y)=y.$$

They say that the operation $(\cdot)$ have: *left* (*right*, *middle*) inverse property [**1, 4**], if

$$\lambda x\cdot xy=y\qquad(\text{respectively,}\quad yx\cdot\rho x=y,\quad x\cdot y=\mu(y\cdot x))$$

for some transformation λ, (resp. ρ, μ) of the set Q.

If the operation $(\cdot)$ in a quasigroup $(Q;\cdot;\overset{\ell}{\cdot};\overset{r}{\cdot})$ has a middle inverse property, then the operations $(\overset{\ell}{\cdot})$ and $(\overset{r}{\cdot})$ have left and right inverse property respectively.

Let $(Q;\circ)$ be a group isotope (i.e. it is isotopic to a group) and let $0\in Q$, then

$$x\circ y=\alpha x+a+\beta y \tag{1}$$

is called a *0-canonical decomposition*, if $(Q;+;0)$ is a group and $\alpha 0=\beta 0=0$. An arbitrary element of a group isotope uniquely defines its canonical decomposition [**2**].

Theorem 1. *Let $(Q;\circ)$ be a group isotope and (1) be its canonical decomposition, then:*

1) *$(\circ)$ has a right inverse property if and only if α an involutive automorphism of $(Q;+)$ and*
$$\alpha a=-a,\qquad \rho=\beta^{-1}JI_a\alpha\beta.$$
2) *$(\circ)$ has a left inverse property with if and only if β an involutive anti-automorphism of $(Q;+)$ and*
$$\beta a=-a,\qquad \lambda=\alpha^{-1}JI_a\beta\alpha,$$
3) *$(\circ)$ is middle inverse property if and only if exist anti-automorphism θ such that*
$$\mu x=\theta x+c,\qquad \theta^2=I_c^{-1},\qquad \alpha=\theta\beta,$$
where $c:=-\theta a+a$.