On the residual of a finite group with seminormal or subnormal factors

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Throughout this paper, all groups are finite and G always denotes a finite group. We use the standard notations and terminology of [**1**]. The monographs [**2**] contain the necessary information of the theory of formations.

If H and K are subgroups of G such that H is permutable with every subgroup of K and K is permutable with every subgroup of H, we say that H and K are mutually permutable. If $G = AB$ and the subgroups A and B are mutually permutable, then G is called a mutually permutable product of A and B, see [**4**, p. 149].

Let $\mathfrak{F}$ be non-empty formation. Then $G^{\mathfrak{F}}$ denotes the $\mathfrak{F}$-residual of G. Obviously, a group G is supersoluble if and only if $G^{\mathfrak{U}} = 1$. Here $\mathfrak{U}$ is the formations of all supersoluble groups. A well-known theorem of Doerk and Hawkes [**2**, Теорема IV.1.18] states that for a formation $\mathfrak{F}$ of soluble groups the $\mathfrak{F}$-residual respects the operation of forming direct products. The above results confirm that the $\mathfrak{F}$-residuals play an important role in the study of the structure of groups. Fortunately, these residuals have a nice behaviour in mutually permutable products.

In [**3, 4**] authors obtained a decomposition $G^{\mathfrak{F}} = A^{\mathfrak{F}}B^{\mathfrak{F}}$ for a group $G = AB$ that is a mutually permutable product of subgroups A and B, where $\mathfrak{F}$ is a saturated formation containing the class $\mathfrak{U}$ of supersoluble groups. The results of Asaad and Shaalan [**5**, Theorem 3.8] and M. Alejandre, A. Ballester-Bolinches, J. Cossey [**6**, Theorem 1] follow from the previous results when A and B are supersoluble.

A subgroup A of a group G is called *seminormal* in G, if there exists a subgroup B such that $G = AB$ and AX is a proper subgroup of G for every proper subgroup X of B. The groups with some seminormal subgroups were investigated in works of many authors, see, for example, the literature in [**7**]. If the subgroups A and B of $G = AB$ are mutually permutable, then A and B are seminormal in G. The converse is false. For instance, $G = [Z_7]Z_6$ is the product of seminormal in G subgroups $A \simeq Z_6$ and $B \simeq [Z_7]Z_2$, but A and B are not mutually permutable. Here Z_n is a cyclic group of order n.

In this paper, we prove the following

Theorem. *Let A and B be seminormal or subnormal subgroups of G and $G = AB$. Let $\mathfrak{F}$ be a saturated formation such that $\mathfrak{U} \subseteq \mathfrak{F}$. If A and B belong to $\mathfrak{F}$, then $G^{\mathfrak{F}} \leq (G')^{\mathfrak{N}}$. Here $\mathfrak{N}$ is the formation of all nilpotent groups.*

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Classes of finite rings

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The concept of formation appeared first in the 1960s in connection with finite solvable groups [**1**]. Further research showed that formations are of general algebraic nature and can be applied to the study of not necessarily solvable finite and infinite groups, Lie algebras, universal algebras and even of a general algebraic system [**2**]. A well-known result in group theory states that any formation of finite groups is saturated iff it is local (see Theorem 4.6 in the book [**3**]). In contrast to the group case, not every saturated formation of Lie and Leibniz algebras, monoids, rings, etc. can be locally defined. However, these formations have found various applications.

Commutative rings have found some interesting applications in cryptography and coding theory. Wedderburn's little theorem says every finite domain is a field. Codes over finite filds, which appear in information and communication theory, have been investigated as images of codes over Galois rings over the ring of integers modulo 4; see [**4**, Chapter 8]. These facts motivate to study properties of finite rings and their classes. In particular it will be interesting to translate some of the most powerful results of the theory of formations of finite groups for formations of rings. A class of rings is a *formation* whenever it contains all homomorphic images of its members and if it is subdirect product closed.

THEOREM 1. [**5**] *The lattice of all formations of finite rings is algebraic and modular.*

Fuzzy sets became applied in fields such as pattern recognition, machine learning and data mining [**6**]. Focusing on the structure of ring, Liu [**7**] introduced and studied the notions of fuzzy subrings and fuzzy ideals, and showed that the images and preimages under onto homomorphisms of fuzzy ideals are fuzzy ideals, but not all the results on rings can be fuzzified.

Let R be a finite commutative ring with an identity element. It is established that there is a one-to-one correspondence between the set of all invariant fuzzy prime ideals of R and the set of all fuzzy prime ideals of each ring of the formation generated by R; see [**5**] for more details.

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