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Can one hear the shape of permutation group?

Vasyl Ustimenko

Missing definitions of the theory of permutation groups and algebraic graph theory such as coherent configuratuion, orbital of permutation group, distance regular and distance transitive graphs, extended bipartite double of graph reader can find in [1], [2] or [3]. Connection of concept of coherent configurations with other objects of algebraic combinatorics are given in [4]. Concepts of Spectral Graph Theory could be found in [5].

Let G be finite group acting faithfully on set Ω. Corresponding group coherent configuration $GCC(G,\Omega)$ is a totality of all invariant binary relations of (G,Ω), i.e subsets of $\Omega\times\Omega$ which are unions of orbitals (G,Ω) (orbits of natural action of G on the set $\Omega\times\Omega$). We say that two non-similar group permutations (G,Ω) and (H,Ω) are isospectral if for each simple graph of symmetric binary relation of $GCC(G,\Omega)$ there exists isospectral graph of $GCC(H,\Omega)$.

Question written in the title of the abstract is about the existence of isospectral pair of permutation groups. We will say that two isospectralpermutation groups (G,Ω) and (H,Ω) are nonisomorphic if group G and H are nonisomorphic. Let $X_n(q)$ be Shevalley group corresponding to Coxeter-Dynkin diagram X_n, $W(X_n)$ be the Coxeter system of Weyl group of root system X_n. Symbols $\Gamma(X_n(q))$ and $\Gamma(W(X_n))$ stand for corresponding Lie and Coxeter geometries. $\Gamma_i(X_n(q))$ and $\Gamma_i(W(X_n))$ are totalities of geometry elements of type corresponding to node i of the diagram. There is natural correspondence between binary relations of $GCC(X_n(q),\Gamma(X_n(q)))$ and $GCC(W(X_n),\Gamma(W(X_n))$. Symbol X stands here for one of characters A,B,C,D. Notice that $GCC(W(B_n),\Gamma(W(B_n)) = GCC(W(C_n),\Gamma(W(C_n))$. Let ϕ be binary relation of $GCC(W(X_n),\Gamma(W(X_n)))$ and $\phi(X,q)$ correspond to ϕ as binary relation in $\Gamma(X_n(q))\times\Gamma(X_n(q))$.

Theorem 1. *If q is odd and $n\geq 3$ then pairs $(B_n(q),\Gamma_n(q))$, $(C_n(q),\Gamma_n(q))$ form families of nonisomorphicisospectral permutation groups.*

Corollary 1. *Let ϕ_i be binary relation of $GCC(W(B_n),\Gamma(W(B_n))$ containing in $\Gamma_i(W(B_n))\times\Gamma_i(W(B_n))$ and $\phi_i(B_n,q)$ and $\phi_i(C_n,q)$are corresponding to ϕ_i invariant binary relationsof $B_n(q)$ and $C_n(q)$ containing in $\Gamma_i(B_n(q))\times\Gamma_i(B_n(q))$ and $\Gamma_i(C_n(q))\times\Gamma_i(C_n(q))$. If q is odd then graphs $\phi_i(B_n,q)$ and $\phi_i(C_n,q)$ are isospectral. If n, $n\geq 3$, i and ϕ_i are fixed then graphs $\phi_i(B_n,q)$ and $\phi_i(C_n,q)$, $q=3,5,7,9,\ldots$ form two families of isospectral regular small world graphs. Their orders grow but diameter is bounded by constant $c(n,j,\phi)$.*

Remark 1. Described above pairs of families of small world graphs are geometrical expanders (see [6]), i. e. families of regular graphs of increasing degree such that the ratio of second largest eigenvalue and degree tends to ∞ when parameter q grows.

Remark 2. We can substitute geometries $B_n(q)$ and $C_n(q)$ for totalities of their flags (cosets by parabolic subgroups).

Theorem 1 demonstrates that we not only unable to hear the shape of graph but unable to distinguish symphonies performed by two orchestras of small world graphs. We unable to distinguish by sound two different families of simple groups of Lie type acting on their geometries.

D. Higman introduced the concept of coherent configurations as a special totality of binary relations on set Ω. Class of coherent configurations contains group coherent configurations. Two nonisomorphic coherent configurations C_1 and C_2 are isospectral if for each simple graph of symmetric binary relation of $GCC(G,\Omega)$ there exists isospectral graph of $GCC(H,\Omega)$.

Homogeneous coherent configurations are generalisations of group configurations of transitive permutation groups. We say that coherent configuration is non-Schurian if it is not of kind $GCC(H,\Omega)$ for some permutation group (H,Ω). Coherent configuration is vertex transitive if its automorphism group acts transitively on Ω. We say that permutation (H,Ω) has a spectral shadow if there is isospectral non-Schurian configuration to $GCC(H,\Omega)$.

Theorem 2. *Permutation groups $(D_n(q),\Gamma_n(q))$, $n \geq 4$ have spectral shadows.*

Remark 3. One of shadows as in theorem 2 is generated by Ustimenko graph $U(n,q)$ (see subject index in [2] and [7]) which is isospectral to dual polar space graph of $D_n(F_q)$ (generator of$GCC(D_n(q),\Gamma_n(q))$. In this case shadow has automorphism group $C_{n-1}(q)$. There is the unique shadow of $(D_n(q),\Gamma_n(q))$ with this automorphism group.

Hemmeter graphs $H(n,q)$ has vertex set $\Gamma'(D_n(q))$ which is a disjoint union of two copies of $\Gamma_n(D_n(q))$. Automorphism group $G(n,q) = D_n(q) \times Z_2$ of this graph acts transitively on this set (see [8]). Hemmeter graph is the so called extended bipartite double of Ustimenko graph (see [2]). Graphs $U(n,q)$ and $H(n,q)$ form families of distance regular but not distance transitive graphs.

Theorem 3. *Let $C(H(n,q))$ be coherent configuration generated by binary relation $H(n,q)$. If q is odd then $C(n,q)$ is a shadow of permutation group $(G(n,q),\Gamma'(D_n(q))$.*

Set of vertices of Double Grassman Graph $J_q(2k+1,k)$ is $V = V(J_q(2k+1,k)) = \Gamma_k(A_{2k+1}(q)) \cup \Gamma_{k+1}(A_{2k+1}(q))$. This graph is simply a restriction of incidence relation of $\Gamma(A_{2k+1}(q))$ on V. The extension $\tilde{A}_{2k+1}(q)$ of $A_{2k+1}(q)$ by contragradientautomorphism of projective geometry acts transitively on V.

Theorem 4. *Permutation group $(\tilde{A}_{2k+1}(q),V)$ has a spectral shadow.*

Remark 4. One of the spectral shadows of theorem 4 is generated by Van Dam distance regular but not a distance transitive graph (see [9]).

Let us consider the largest Schubert cell $Sch(X_n(q))$ of geometry $\Gamma(X_n(q))$ on $\Gamma_n(X_n(q))$, i. e. the largest orbit of Borel subgroup of $X_n(q)$ on this set. Let $C(X_n(q))$ be non-Schurian coherent configuration generated by the restrictions of orbitals of$(X_n(q),\Gamma_n(X_n(q))$. The coherent configuration $C(B_3(q))$ contains binary relation of Brower graph $S(q)$ [10] which is distance regular, but not distance transitive one. In fact it is a restriction of binary relation of dual polar graph $D_4(F_q)$. Borel subgroup B of $B_3(q)$ is an automorphism group of $C(B_3(q))$ which acts transitively on set of vertices of $S(q)$. The extended bipartite double of this graph is another distance regular but not a distance transitive graph [10] which is known as Pasechnik graph $S'(q)$ with vertex set $Sch(B_3(q)) \cup Sch(B_3(q)$. Subgroup of the automorphism group of $S'(q)$ isomorphic to $B \times Z_2$ acts transitively on vertex set of this graph. Let $C(S'(q))$ be non-Schurian coherent configuration generated by $S'(q)$.

Theorem 5. *If q is odd then non-Schurian vertex-transitive coherent configurations $C(B_3)(q)$ and $C(C_3(q))$ are isospectral.*

Remark 5. Let $Z(q)$ be a cospectral graph to $S(q)$ in coherent configuration $C(C_3(q))$. Then $Z(q)$ is a new distance regular but not a distance transitive graph.

Let $Z'(q)$ be extended bipartite double graph of $Z(q)$. We consider coherent configuration $C(Z'(q))$ generated by $Z'(q)$.

Theorem 6. *If q is odd then non-Schurian vertex transitive coherent configurations $C(S'(q))$ and $C(Z'(q))$ are isospectral.*

Diameter d of isospectral graphs $G(q)$, $G^*(q)$, where $G(q)$ is one of the graphs $S(q)$, $S'(q)$, $U(n,q)$, $H(n,q)$, $VD(n,q)$ and $G^*(q)$ and is one of the graphs $Z(q)$, $Z'(q)$, $D_n(q)$, $D'_n(q)$, $Jq(2k+1,2k)$ is independent from q constant. Let $G_J(q)$ and $G^*{}_J(q)$ (J is nonempty proper subset of $\{1,2,\ldots,d\}$) be graphs of binary relation „two vertices of $G(q)$ ($G^*(q)$) are at distance $s \in J$". Assume that parameters n and k are fixed and odd q runs via $3,5,7,9,\ldots$. Then pairs of isospectral graphs $(G_J(q), G*_J(q))$ form a family of isospectral distance-regular and a family of small word graphs which are geometrical expanders. Families of small world graphs, families of distance regular graphs and families of geometrical expanders have many applications in Computer Science, Natural Sciences and corresponding technologies. Some analogue of theorems 1-6 for infinite fields are formulated in [11], where techniques of Voronoi diagrams for graphs [12] and corresponding Voronoi cells are used.

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This research note is dedicated to the memory of the prominent algebraist Volodymyr Vasyliovych Kirichenko who made an outstanding impact on the development of Mathematics in Kyiv State University as well as the development of Algebra Research in Ukraine.