Properties of quotient modules composed of annihilator modules in Steenrod algebra

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This work studies structures of modules $B(n) = (A(n-1)/A(n))^*$ dual to $A(n-1)/A(n)$ as well as $A(n)^*$, $A(n)^+/A(n-1)^+$, $(A(n-1)/A(n))^*$, $A(n-1)^*/A(n)^*$ where $A(n)$ are annihilators **[1]** , all members of Steenrod algebra such that they null all cohomology elements from cohomology classes with degree less or equal then n and $A(n)^+ \in A^*$ is composed from annihilators of $A(n)$ module from the dual Steenrod algebra A^*. Result is stated in

Theorem 1. *(1) Annihilator $A(n)^+$ of module $A(n)$ is A^*-comodule and*

$$A(n)^+ \cong (A/A(n))^*$$

(2) $A(n)^+$ is generated by all monomials of multiplication less or equal then n. $A(n)^+$ is induced A^*-comodule and

$$A(n)^* \cong A^*/A(n)^+$$

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(3) $(A(n-1)/A(n))^*$ is a left induced A^*-comodule and as a vector space over Z/p has a basis generated by all monomials of multiplication n in A^*. There are isomorphisms:

$$(A(n-1)/A(n))^* \cong A(n)^+/A(n-1)^+ \cong A(n-1)^*/A(n)^*$$

Theorem 2. *(1) $B(n)$ is a graded Hopf comodule over Steenrod algebra A^* with coproduct $\phi_n^* : B(n) \to A^* \bigotimes B(n)$, $\phi_n^*([\alpha]) = \sum_i \alpha_i' \bigotimes [\alpha_i'']$ induced by coproduct in comodule $A(n)^+$, with homomorphism property*

$$\phi_n^*([\alpha]*[\beta]) = \phi_n^*([\alpha\beta]) = (\psi^* \otimes \psi_n^*)(Id_{A^*} \otimes T \otimes Id_{B(n_2)})(\phi_{n_1}^*([\alpha]) \otimes \phi_{n_2}^*([\beta])) \stackrel{def}{=} \phi_{n_1}^*([\alpha]) * \phi_{n_2}^*([\beta])$$

where $\psi_{n_1+n_2}^* : B(n_1) \bigotimes B(n_2) \to B(n_1+n_2)$ is a product defined by $\psi_{n_1+n_2}^*([\alpha] \otimes [\beta]) = [\psi^*(\alpha \otimes \beta)] = [\alpha\beta] = [\alpha] * [\beta]$ induced by product ψ^* in A^*, $n = n_1 + n_2$, T is a transposition, $[\alpha]$ in $B(n_1)$, and $[\beta]$ in $B(n_2)$.

(2) $B(n) = \bigoplus_s B(n)^s$ is the direct sum of Hopf comodules

$$B(n)^s = \{\tau_0^{s_0}\tau_1^{s_1}\tau_2^{s_2} \ldots \xi_1^{r_1}\xi_2^{r_2}\xi_3^{r_3} \ldots \in A^* | \sum_i s_i = s \ , \ n = \sum_i s_i + 2\sum_i r_i\}$$

(3) $B(n)_t = \bigoplus_s B(n)_t^s$ is the direct sum of comodules $B(n)_t = (A(n)^+ \cap A_t^*)/(A(n-1)^+ \cap A_t^*)$ defined on the filtration of dual Steenrod algebra A^* by Hopf subalgebras

$$A_{-1}^* \subset A_0^* \subset A_1^* \subset \ldots \subset A_n^* \subset A_{n+1}^* \subset \ldots A^*$$

where $A_t^* = Z_p\{\xi_1, \xi_2, \ldots\} \bigotimes E\{\tau_0, \tau_1 \ldots \tau_t\}$ and $A_{-1}^* = Z_p\{\xi_1, \xi_2 \ldots\}$. The restrictions of the coprogduct and product (1) on the filtration are well defined maps: $\phi_{n,t}^* : B(n)_t \to A^* \otimes B(n)_t$ and $\psi_{n_1 n_2,t}^* : B(n_1)_t \otimes B(n_2)_t \to B(n_1+n_2)_t$.

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Finite groups with given properties of normalizers of Sylow subgroups

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We consider only finite groups. We use notations and definitions from [**1**].

Let $\mathfrak{F}$ be a non-empty formation. A subgroup H is called $\mathfrak{F}$-subnormal in G, if either $H = G$, or there exists a maximal chain of subgroups $H = H_0 \le H_1 \le \cdots \le H_{n-1} \le H_n = G$ such that $H_i^{\mathfrak{F}} \le H_{i-1}$ for $i = 1, \dots, n$.

Recall that the class of groups $\mathrm{w}^*\mathfrak{F}$ is defined as follows:

$\mathrm{w}^*\mathfrak{F} = (G \mid \pi(G) \subseteq \pi(\mathfrak{F})$ and every normalizer of Sylow subgroup of G is $\mathfrak{F}$-subnormal in $G)$.

Theorem 1. *Let $\mathfrak{F}$ be a non-empty hereditary formation. Then the following statements are true.*

(1) $\mathfrak{F} \subseteq \mathrm{w}^*\mathfrak{F}$.
(2) $\mathrm{w}^*\mathfrak{F} = \mathrm{w}^*(\mathrm{w}^*\mathfrak{F})$.
(3) *If a formation* $\mathfrak{F}_1 \subseteq \mathfrak{F}$ *then* $\mathrm{w}^*\mathfrak{F}_1 \subseteq \mathrm{w}^*\mathfrak{F}$.
(4) $\mathrm{w}^*\mathfrak{F}$ *is a formation and from* $G \in \mathfrak{F}$ *it follows that every Hall subgroup of* G *belongs to* $\mathfrak{F}$.

According to [**2**], the arithmetic length of a soluble group G is defined as $\max\{l_p(G)\}$, where $l_p(G)$ is p-length of the group G for all $p \in \pi(G)$. Note that the class $\mathfrak{L}_a(1)$ of all soluble groups whose arithmetic length ≤ 1 is a hereditary saturated Fitting formation.

Theorem 2. *Let $\mathfrak{F}$ be a hereditary saturated formation and $\mathfrak{F} \subseteq \mathfrak{L}_a(1)$. Then $\mathrm{w}^*\mathfrak{F} = \mathfrak{F}$.*

Corollary 1. (1) [**3**] *If $\mathfrak{N}^2$ is the class of all metanilpotent groups, then $\mathrm{w}^*\mathfrak{N}^2 = \mathfrak{N}^2$.*

(2) [**3**] *If $\mathfrak{N}\mathfrak{A}$ is the class of all groups G with the nilpotent commutator subgroup G', then $\mathrm{w}^*\mathfrak{N}\mathfrak{A} = \mathfrak{N}\mathfrak{A}$.*

(3) $\mathrm{w}^*\mathfrak{L}_a(1) = \mathfrak{L}_a(1)$.

We note that $\mathrm{w}^*\mathfrak{N}^3 \neq \mathfrak{N}^3$.

References

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