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J-Noetherian Bezout domain which are not of stable range 1

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All rings considered will be commutative and have identity.

A ring R is a ring of stable range 1 if for any $a, b \in R$ such that $aR + bR = R$ we have $(a + bt)R = R$ for some $t \in R$.

An element a is an element of stable range 1 if for any $b \in R$ such that $aR + bR = R$ we have $a + bt$ is an invertible element for some $t \in R$.

An element $a \in R$ is an element of almost stable range 1 if R/aR is a ring of stable range 1.

By a Bezout ring we mean a ring in which all finitely generated ideals are principal.

By a J-ideal of R we mean an intersection of maximal ideals of R.

A ring R is J-Noetherian provided R has maximum condition of J-ideals.

A commutative ring R is called an elementary divisor ring [**3**] if for an arbitrary matrix A of order $n \times m$ over R there exist invertible matrices $P \in GL_n(R)$ and $Q \in GL_m(R)$ such that

$PAQ = D$ is diagonal matrix, $D = (d_{ii})$,

$d_{i+1,i+1}R \subset d_{ii}R$.

Let R be a Bezout domain. An element $a \in R$ is called a neat element if for every elements $b, c \in R$ such that $bR + cR = R$ there exist $r, s \in R$ such that $a = rs$ where $rR + bR = R$, $sR + cR = R$ and $rR + sR = R$. A Bezout domain is said to be of neat range 1 if for any $c, b \in R$ such that $cR + bR = R$ there exists $t \in R$ such that $a + bt$ is a neat element.

Theorem 1. *A commutative Bezout domain R is an elementary divisor domain if and only if R is a ring of neat range 1.*

Theorem 2. *A nonunit divisor of a neat element of a commutative Bezout domain is a neat element.*

Theorem 3. *Let R be a J-Noetherian Bezout domain which is not a ring of stable range 1. Then in R there exists an element $a \in R$ such that R/aR is a local ring.*

By [**8**], any adequate element of a commutative Bezout ring is a neat element. An element a of a domain R is said to be adequate, if for every element $b \in R$ there exist elements $r, s \in R$ such that (1) $a = rs$; (2) $rR + bR = R$ (3) $\hat{s}R + bR \neq R$ for any $\hat{s} \in R$ such that $sR \subset \hat{s}R \neq R$. A domain R is called adequate if every nonzero element of R is adequate [**4**].

Theorem 4. *Let R be a commutative Bezout element and a is non-zero nonunit element of R. If R/aR is local ring, then a is an adequate element.*

Theorem 5. *Let R be a J-Noetherian Bezout domain which is not a ring of stable range 1. Then in R there exists a nonunit adequate element.*

Theorem 6. *Let R be a Bezout domain in which every nonzero nonunit element has only finitely many prime ideals minimal over it. Then the factor ring R/aR is the finite direct sum of valuation rings.*

Let R be a domain and $a \in R$. Denote by *minspec* a the set of prime ideals minimal over a.

THEOREM 7. *Let R be a commutative Bezout domain in which any nonzero prime ideal is contained in a finite set of maximal ideals. Then for any nonzero element $a \in R$ such that the set minspec (aR) is finite, the factor ring $\overline{R} = R/aR$ is a finite direct sum of semilocal rings.*

THEOREM 8. *Let R be a Bezout domain in which every nonzero nonunit element has only finitely many prime ideals minimal over it. Then the factor ring R/aR is everywhere adequate if and only if R is a finite direct sum of valuation rings.*

THEOREM 9. *Let R be a Bezout domain in which every nonzero nonunit element has only finitely many prime ideals minimal over it and any nonzero prime ideal spec (aR) is contained in a finite set of maximal ideals. Then a is an element of almost stable range 1.*

Open Question. Is it true that every commutative Bezout domain in which any non-zero prime ideal is contained in a finite set of maximal ideals is an elementary divisor ring?

References

1. D. Estes and J. Ohm, *Stable range in commutative rings*, J. Alg. **7** (1967) 343–362.
2. M. Henriksen, *Some remarks about elementary divisor rings*, Michigan Math. J. **3** (1955/56) 159–163.
3. I. Kaplansky, *Elementary divisors and modules*, Trans. Amer. Math. Soc. **66** (1949) 464–491.
4. M. Larsen, W. Levis and T. Shores, *Elementary divisor rings and finitely presented modules*, Trans. Amer. Math. Soc. **187** (1974) 231–248.
5. S. McAdam and R. Swan, Unique comaximal factorization, J. Alg. **276** (2004) 180–192.
6. W. McGovern, *Bezout rings with almost stable range 1 are elementary divisor rings*, J. Pure Appl. Alg. **212** (2007) 340–348.
7. B. V. Zabavsky, *Conditions for stable range of an elementary divisor rings*, Comm. Alg. **45** (2017), no. 9, 4062–4066.
8. B. V. Zabavsky, *Diagonal reduction of matrices over finite stable range rings*, Mat. Stud. **41** (2014) 101–108.

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