Contact information

**Mykola Pratsiovytyi**
Institute of Mathematics of NASU, National Pedagogical Dragomanov University, Kyiv, Ukraine
*Email address*: prats4444@gmail.com

*Key words and phrases.* Fractal, self-affine set, function with fractal properties, transformation preserving fractal dimension, singular probability distribution, system of encoding (representation) of real numbers.

# Mathematical structures with fractal properties in the space of sequences of zeros and ones

Mykola Pratsiovytyi, Sofiia Ratushniak

Let $A=\{0,1\}$ be an alphabet and let $L\equiv A\times A\times A\times ...\times A\times ...$ be a space of sequences of zeros and ones. An element of this space is denoted by a $(a_n)$ or $\overline{a}$.

A mapping $f$ of space $L$ onto a set $[0,1]$ given by a formula

$$L\ni(\alpha_n)\xrightarrow{f}x=\frac{\alpha_1}{2}+\frac{\alpha_2}{2^2}+...+\frac{\alpha_n}{2^n}+...\equiv\Delta^2_{\alpha_2\alpha_2...\alpha_n...}$$

is called a classical binary representation.

The binary representation of numbers is a useful tool to develop binary analysis, metric and probabilistic theory of numbers. The simplicity of geometry of this representation generates various applications in the theory of fractals. It is instrument for studying functions with various fractal properties and complicated local structure (behaviour).

The metrization of space $L$ by means of binary representation and the function

$$f_1(\overline{x},\overline{y})=\sum_{k=1}^{\infty}\frac{|x_k-y_k|}{2^k}$$

allows us to develop a theory of fractals in the space $L$, which has own features and differences as compared with the Euclidean metric in $[0,1]$.

Let $E\subset L$ be a set of all sequences such that frequencies of digits 0 and 1 are equal to $p_0\neq 0$ and $p_1=1-p_0\neq 0$ respectively. In this set, we consider the metric

$$f_2(\overline{p},\overline{q})=|\ln\frac{p(1-q)}{q(1-p)}|,$$

where $\overline{p}\equiv(p,1-p)$, $\overline{q}\equiv(q,1-q)$, and fractal subsets defined by different conditions.

In the metric space $(L,f_2)$ we develop theory of fractal sets and others objects with fractal properties.

In our talk we develop fractal analysis in the space $L$ by using different two-symbol representations of real numbers, which are generalizations of classical binary representation ($Q_2^*$-representation) or its analogues ($A_2$-continued fraction representation, etc.). Self-similar and non-self-similar representations with zero and non-zero redundancy are among them.

Functions with fractal properties and dynamical systems with locally complicated mappings are studied in detail. In particular, we discuss properties of distribution of values of function $f_\varphi(\Delta^{Q_2^*}_{\alpha_1\alpha_2...\alpha_n...})=\Delta^{Q_2^*}_{\varphi_1(\alpha_1,\alpha_2)\varphi_2(\alpha_2,\alpha_3)\varphi_n(\alpha_n,\alpha_{n+1})...}$, where $(\varphi_n)$ is a sequence of finite functions that are defined on a four-element set $A^2\equiv A\times A$ and take values from the set $A$; and $\Delta^{Q_2^*}_{\alpha_1\alpha_2...\alpha_n...}$ is the $Q_2^*$-representation of number $x\in[0,1]$.

Contact information

**Mykola Pratsiovytyi**
Institute of Mathematics of NASU, National Pedagogical Dragomanov University, Kyiv, Ukraine
*Email address*: prats4444@gmail.com

**Sofiia Ratushniak**
Institute of Mathematics of NASU, Kyiv, Ukraine
*Email address*: ratush404@gmail.com

*Key words and phrases.* The space of sequences of zeros and ones, non-standard metric, fractal set, function with fractal properties, fractal dimension.

# On similarity of tuples of matrices over a field

Volodymyr Prokip

Let $\mathbb{F}$ be a field. Denote by $\mathbb{F}_{m\times n}$ the set of $m \times n$ matrices over $\mathbb{F}$ and by $\mathbb{F}_{m\times n}[x_1, x_2, , x_n]$ the set of $m \times n$ matrices over the polynomial ring $\mathbb{F}[x_1, x_2, , x_n]$. In what follows, we denote by $I_n$ the $n \times n$ identity matrix and by $0_{n,k}$ the zero $m \times n$ matrix. The Kronecker product of matrices $A = \left[\ a_{ij}\ \right] \in \mathbb{F}_{m\times n}$ and $B$ is denoted by $A \otimes B = \left[\ a_{ij}B\ \right]$.

Two tuples of $n \times n$ matrices $\mathbf{A} = \{A_1, A_2, \ldots, A_k\}$ and $\mathbf{B} = \{B_1, B_2, \ldots, B_k\}$ over a field $\mathbb{F}$ are said to be simultaneously similar if there exists a nonsingular matrix $U \in \mathbb{F}_{n\times n}$ such that $A_i = U^{-1}B_iU$ for all $i = 1, 2, \ldots, k$. The task of classifying square matrices up to similarity is one of the core and oldest problems in linear algebra (see [**1**], [**2**], [**3**] and references therein), and it is generally acknowledged that it is also one of the most hopeless problems already for $k = 2$. For given matrices $A_i, B_i \in \mathbb{F}_{n\times n}$ we define matrices

$$M_i = \left[\ A_i \otimes I_n - I_n \otimes B_i^T\ \right] \in \mathbb{F}_{n^2\times n^2},\ i = 1, 2, \ldots, k; \quad \text{and} \quad M = \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_k \end{bmatrix} \in \mathbb{F}_{kn^2\times n^2}.$$

Theorem 1. *If two tuples of $n\times n$ matrices $\mathbf{A} = \{A_1, A_2, \ldots, A_k\}$ and $\mathbf{B} = \{B_1, B_2, \ldots, B_k\}$ over a field $\mathbb{F}$ are simultaneously similar then* $\operatorname{rank} M < n^2$.

Let $\operatorname{rank} M = n^2 - r$, where $r \in \mathbb{N}$. For the matrix $M$ there exists a nonsingular matrix $U \in \mathbb{F}_{n^2\times n^2}$ such that $MU = \left[\ H \quad 0_{kn^2,r}\ \right]$, where $H \in \mathbb{F}_{kn^2\times(n^2-r)}$. Put $U = \left[\ U_1 \quad U_2\ \right]$, where $U_2 \in \mathbb{F}_{n^2\times r}$. For independent variables $x_1, x_2, \ldots, x_r$ we construct the vector

$$U_2 \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{bmatrix} = \begin{bmatrix} V_1(\overline{x}) \\ V_2(\overline{x}) \\ \vdots \\ V_n(\overline{x}) \end{bmatrix}, \quad \text{where} \quad V_i(\overline{x}) = V_i(x_1, \ldots, x_r) \in \mathbb{F}_{n,1}[x_1, x_2, \ldots, x_r].$$

Theorem 2. *Two tuples of $n\times n$ matrices $\mathbf{A} = \{A_1, A_2, \ldots, A_k\}$ and $\mathbf{B} = \{B_1, B_2, \ldots, B_k\}$ over a field $\mathbb{F}$ of characteristic 0 are simultaneously similar if and only if the matrix* $\begin{bmatrix} V_1^T(\overline{x}) \\ V_2^T(\overline{x}) \\ \cdots \\ V_n^T(\overline{x}) \end{bmatrix}$ *is nonsingular.*